MASTER

ACTUARIAL SCIENCE

MASTER'S FINAL WORK

DISSERTATION

LIFE ANNUITIES AND RUIN

JOANA CATALINA MENDES MOREIRA SAÚDE GREGÓRIO

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SUPERVISION

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Abstract

This work intends to combine two major topics under the actuarial framework: life annuities and ruin theory, as to determine the probability of financial ruin for life annuities' portfolios.

Two main perspectives may be considered: the household's and the life insurance company's, for which different models apply. Time constraints and limitations on text length became the reason why only the company's perspective has been explored, using a classic individual risk model.

After an extensive literature review the basics on life annuities and ruin theory are explained and a case study is toiled. Firstly, the theoretical framework is developed, with a useful result, not found in the literature, being obtained; and finally, the application follows.

The problem to be solved consists broadly in studying whether reserves are high enough to keep the ruin probability under control, when considering a given insurer's portfolio of life annuities, divided into homogeneous groups. This is done in two different ways: computing the ruin probabilities, given the initial reserve; and finding the initial reserves' allocation amongst the groups that maximizes the survival probabilities. Frostig and Denuit (2009) is the main reference. Some significant results are observed.

Key words: drawdown analytics; life annuities; individual and collective risk models; lifetime ruin probability; lifetime survival probability maximization.

Resumo

Este trabalho pretende combinar dois grandes tópicos num contexto atuarial: rendas contingentes sobre a vida humana e teoria da ruína, de forma a determinar a probabilidade de ruína financeira para carteiras de anuidades-vida.

Duas principais perspetivas podem ser consideradas nesta situação: a dos indivíduos e a das seguradoras de vida, com aplicação de diferentes modelos. Limitações de tempo disponível e extensão do texto conduziram a que apenas a perspetiva das empresas fosse objeto de estudo, aplicando-se o modelo de risco individual clássico.

Após uma extensiva revisão literária, os conceitos fundamentais sobre anuidades-vida e teoria da ruína são explicados e um caso de estudo é tratado. Primeiramente, os conceitos teóricos são desenvolvidos, de tal forma que um resultado, não encontrado na literatura, é obtido; segue-se a aplicação dos conceitos a uma carteira de riscos real.

O problema a ser resolvido consiste em determinar se as reservas são suficientes para manter a probabilidade de ruína sob controlo, quando considerando tal carteira de anuidades-vida, dividida em grupos homogéneos. Dois procedimentos são seguidos: calcular as probabilidades de ruína, a partir de uma reserva inicial; e encontrar a melhor alocação das reservas iniciais pelos grupos de forma a maximizar as probabilidades de sobrevivência. Frostig e Denuit (2009) é a principal referência bibliográfica. Alguns resultados significativos são observados.

Palavras-chave: análise de utilização de riqueza; anuidades-vida; modelos de risco individual e coletivo; probabilidade de ruína; maximização da probabilidade de sobrevivência.

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Glossary

BoE – Bank of England

DB – Defined Benefit

DC – Defined Contribution

cdf – Cumulative Distribution Function

cf. - confer

CLT – Central Limit Theorem

CMI – Continuous Mortality Investigation

CRM – Collective Risk Model

EIAs – Equity-Indexed Annuities

EPV(s) – Expected Present Value(s)

FR – Fixed Rate

IAN – International Actuarial Notation

IFoA – Institute and Faculty of Actuaries

IRM – Individual Risk Model

ISSA – International Social Security Association

mu – Monetary Unit

pa – per Annum

pdf – Probability Density Function

PoR – Probability of Ruin

PV – Present Value

rv(s) – Random Variable(s)

S1PFA – SAPS 1 Pensioners Females Amounts

S1PMA – SAPS 1 Pensioners Males Amounts

SER – Single Equivalent Rate

TSIR - Term Structure of Interest Rates

UDD – Uniform Distribution of Deaths

USA – United States of America

UK - United Kingdom

Chapter 1 - Introduction

1.1 Motivation

Following the studies carried out during the Actuarial Science Master's program, a particular combination seemed quite interesting: life annuities, a financial mathematics topic, and financial ruin, a risk theory topic. The main purpose of this work is to progress in the knowledge of these two topics taken together. This first chapter serves as an introductory note to the object and motivation.

1.1.1 Companies and households

When one speaks about life annuities or financial ruin, basically the question is to what branch of economic and social life they are applied and under whose perspective the analysis is conducted: the life insurer perspective or the household perspective?

Life insurance companies, pension funds and social security are naturally concerned about financial sustainability when providing for retirement benefits. For this reason, the main research and models developed, as explained below in the literature review, are in line with actuarial companies' needs and restraints, which are to cope with a wide range of specific requirements. These classic risk models, meant to deal with portfolios and more or less standard benefits, can be developed under a collective or an individual approach. This distinction though is not connected to the individual needs and specifications of policyholders. Rather, it refers to the fact that claims may be considered as a whole or individually. And so models may assume certain distributions considering such division.

On the other hand, individual households have always been concerned about financial sustainability throughout retirement and so it is important to set strategies as to allocate funds to the most appropriate financial instruments and guarantee a desired consumption level. The continued improvements in mortality make the problem more and more serious.

An important fact is then that risk models, particularly those applicable to life annuities and ruin, may be constructed essentially from two perspectives: a company's and a household's. For the company, the classic risk models apply, with distinction between collective and individual, and with a specific goal: set strategies to allocate funds to the most appropriate financial instruments as to guarantee the benefits of a whole portfolio of risks. For the household, meant as the policyholder, the annuitant, the member of a pension fund, more recent and in development models exist. These models have a higher degree of complexity because specific needs are harder to meet than those of a larger group of risks, in which there is a margin for generalization.

1.1.2 Income drawdown options for households

An "income drawdown option" (MacDonald *et al.*, 2013) consists of determining a strategy so that funds accumulated until retirement are not exhausted before death occurs, but don't last for much more than death either, especially in the case when there are no bequest goals.

In most countries (*cf. ISSA*'s website), people in active status contribute regularly to the social security with part of their earnings, in hopes of obtaining a steady income throughout their retirement. However, since this state pension scheme is usually a non-funded one, where income is directly used to pay up due pensions, state pensions are often small and subject to cuts, depending on the state's financial health. For instance, the Portuguese pension scheme, which is basically a pay-as-you-go system, is revised whenever the Government decides it is necessary to do so. For this reason, additional forms to guarantee revenue for retirement exist: pension plans, usually provided by the employer; assurances, in the form of lump sums or life annuities; or a combination of invested assets, producing enough returns to finance old age.

MacDonald *et al.* (2013) have carried out an extensive study on peoples' attitudes towards retirement, based on the experience from the most developed countries, such as the *UK*, the *USA* and Canada. This study offers a literature review on drawdown analytics, providing information about "how people do, could and should drawdown their financial savings". Four main arguments follow.

1. There are several risks, as well as advantages and disadvantages, associated with income drawdown options which are well worth exploring. In sum, these include

longevity, liquidity, annuity price, market, financial investment plus misinformation and mortality risks.

2. The problem is that people are not well informed about retirement drawdown options and, as a result, are not enough confident. In fact, even those who are informed are usually not very confident regarding the financial intermediary, the insurance company or pension fund which provides the services. There is a generalized fear that default from service providers will affect the income bought and that Governments are not able to protect and guarantee all the benefits. For instance, the cases of Lehman Brothers, Conseco or Enron are familiar to all. These companies went bankrupt, leaving their employees and policyholders unprotected, with financial ruin affecting families and firms alike.
3. The solution is financial advisement in retirement. It is necessary to promote awareness amongst retirees and potential retirees about the available options and how these adapt to their specific needs. This would also affect companies in a positive way, because if insurers know what policyholders need and which products are suitable, then they'll be able to develop business in a more efficient way.
4. This analysis makes sense only in certain countries and, particularly, certain segments of society. There has to be an established market for insurance and/or pension funds, as well as investments and personal accounts, and since these are all expensive affairs, there must be adequate demand to sustain the market, complemented with appropriate supply. The individuals participating in these markets should therefore be wealthy enough to even consider accumulating wealth in a way that current consumption is being fulfilled as well.

Though life annuities have been a reality for hundreds of years, only in recent decades and in some countries they have been considered as an alternative to social security, *cf.* Poterba (1997). Most people when facing the decision of choosing their mean of income for retirement choose a lump sum instead of an annuity, as shown by several surveys on health and retirement in the major annuity markets (*UK*, *USA* and Canada), see Helman and Christie (2012) and David Batty *et al.* (2014).

1.2 Literature Review

1.2.1 The insurer perspective

Most literature concerning life annuities studies under a company's perspective is centered in the study of *DB*, *DC* or hybrid (which is a combination of the previous ones) pension plans. The difference between these main two types of pension plans is the fact that *DB* plans provide a defined benefit throughout retirement, usually by the means of an annuity, whereas *DC* plans provide a balance account amount at retirement subject to investment performance over the accumulation phase, usually by the means of a lump sum. However, it is also possible to annuitize via an insurance company and several risk models, under an insurance perspective, have been developed and could eventually be applied. Frostig and Denuit (2009) was the only paper found under this perspective. Most papers develop the insured perspective within the insurance company and the best products to choose.

1.2.2 The policyholder perspective

In this subsection the main papers involving the optimization of annuitization time and amount as to avoid ruin, under a household's perspective, are presented. Note that annuitization refers to substituting one's wealth by an annuity, whose payments may come from several sources.

As a first reference to annuitization studies, Yaari (1965) proved that, in the absence of bequest motives and in a deterministic financial economy, consumers would annuitize all of their wealth. Richard (1975) generalized this result to a stochastic financial environment, and Davidoff, Brown and Diamond (2003) established the robustness of Yaari's result. The common assumption of these articles is a rational utility-maximizing economy, in which individuals present rigid inter-temporal preferences and pre-determined relative risk aversion, a von Neumann and Morgenstern framework (von Neumann and Morgenstern, 1953) , which is difficult to apply in practice, since it does not reflect reality adequately.

Other papers focused on the risk and portfolio management, applying the principle of probability maximization to achieve certain goals, e.g. utility maximization. Browne (1995, 1999, 1999a, 1999b) derived optimal dynamic strategies to minimize the probability of shortfall, firstly mixing utility-maximization and shortfall-minimization and afterwards finding optimal wealth allocation and ruin probability minimization considering investments and returns. Milevsky, Ho and Robinson (1997) studied the importance of avoiding outliving wealth as a first approach to minimize the probability of lifetime ruin or shortfall, described as "to minimize the probability of running out of money before the (uncertain) date of death" (page 54, lines 8-9). In this paper, the authors enhanced the need to match low and high risk, low and high return investments with the correct life timing as to best serve one's needs, considering annual consumption and eventually bequest.

Milevsky and Robinson (2000) used the Lifetime *PoR* and the Eventual *PoR* as risk measures for retirees in a static environment. Lifetime *PoR* is the probability that net wealth, meant as the wealth after all expenses are paid, is exhausted before the stochastically computed time of death is reached. Eventual *PoR* is the probability that net wealth is exhausted anytime in the future as if considering an infinite lifetime.

Milevsky (2001) then studied the options regarding annuitization with the goal of determining its benefits and returns and find the optimal policies for a specific situation. Young (2004) extended Milevsky and Robinson (2000) and Milevsky (2001) works to a stochastic environment, studying the optimal investment strategy to minimize the lifetime ruin probability for an individual who consumes at a specific constant rate or proportion of wealth and invests in a complete financial market (Björk 2004), in risky and riskless assets, but without the ability to buy annuities. Using a continuous-time framework and optimal stochastic control, this work differs from the above since it finds the optimal dynamic investment strategy to minimize the probability of lifetime ruin, instead of computing the probability of ruin coming from a set of fixed investment strategies. As a follow up work, Milevsky, Moore and Young (2006) allowed the policyholder to purchase immediate life annuities, determining, in addition, the optimal annuity purchase strategy. Using an optimal stopping model (Gheorghe Oprisan *et al.*, 2010), they concluded that the individual will only annuitize if there is enough wealth to guarantee in full a certain level of desired consumption for life. In this case, all wealth available is annuitized. Bayraktar and Young (2007) assumed the individual could purchase a deferred annuity instead. The authors found that he/she will only annuitize in the same situation as above, provided that wealth is enough to sustain life during the deferral period.

Finally, Wang and Young (2011) considered that the individual can buy commutable immediate life annuities, with surrender proportion and timing at his/her discretion. In this case, there is the additional need to determine an optimal surrender strategy. This alters the market assumption, passing from a complete to an incomplete market, due to the introduction of the proportional surrender charge. Wang and Young (2011) found that if the surrender charge is low enough, then the individual is pushed to annuitize partially instead of only fully, if at all, as in the previous cases. The flexibility in annuities might be such an important factor as people might perceive them as more adaptable means of sustaining retirement.

1.3 Organization of the Text

In this work, after the comprehensive investigation of the literature on the subject, the insurer perspective, meant to deal with portfolios and more or less standard benefits, was chosen. The household point of view, although appealing, could not be embraced under the existing constraints on time and work dimension.

Using a real portfolio of life annuities from a certain life insurance company, the thesis will focus on how the portfolio might lead the life insurance company to financial ruin. The paper developed by Frostig and Denuit (2009), being of a particular interest to the purpose of the work, will be closely followed.

The outline of the text is as follows. In Chapter 2 the basics on life annuities and ruin theory are explained. Chapters 3 and 4 deal with the case study under, respectively, a theoretical and a practical point of view: on one hand, Chapter 3 exhibits the theoretical results that could be potentially useful to address the problem, and a useful auxiliary result not found in the searched literature; on the other hand, Chapter 4 presents the results obtained when the theory is applied to the real portfolio of life annuities. Finally, Chapter 5 gives the conclusions and final thoughts, i. e. findings and future research.

Chapter 2 - Basics on Life Annuities and Ruin Theory

2.1 Basics on Life Annuities

The following concepts may be found in introductory books on Financial Mathematics and Actuarial Mathematics, such as Broverman (2010) and Dickson *et al.* (2009).

An annuity is a series of payments made at specified points in time during a specified period until maturity. At maturity the payments stop. There are also annuities with no maturity, i. e., whose payments continue forever, called perpetuities. Recall that annuitization refers to substituting one's wealth by an annuity (Chapter 1).

Annuities may be classified into different categories according to the type and moments of the payments. The major categorization for annuities distinguishes between certain and uncertain. Annuities certain are those whose payments are certain to happen at the specified payment dates, i. e., have an occurrence probability equal to 1. On the other hand, uncertain annuities are those whose payments are not certain to happen at the specified payment dates, rather being contingent on the occurrence of an external event and thus having an occurrence probability less than 1. Life annuities, one of the main topics to be studied in this paper, are uncertain annuities, since they are contingent on the statuses of lives.

Life annuities may depend on the status of only one life or more than one life and the payments may be dependent on death, survival or even health status. In most cases, payments are level, but they may be variable (increasing, decreasing or even non monotonous). Usually life annuities are evaluated annually, over parts of a year, say quarters, months, even days, or continuously. In fact, the payments under annuities might be made at any regular interval of time, but these are the most common. Any other time scale may be obtained as an extension of the analysis below. In practice, most actuarial software evaluate benefits using continuous life annuities, for the sake of computation simplicity, and, if necessary, adjust the result to account for the fact that annuities aren't in reality paid continuously.

As far as annuities are concerned, there are mainly two important quantities: the *PV rv* and the *EPV* or actuarial value, which is used to compute the annuity's price, the basic premium for a life insurance contract or the contribution rate or expense to account for a pension plan. In the following of this section these are the two quantities considered. It is also important to refer at this point that life annuities depend/are established on/under two bases: a demographic basis, which deals with the demographic assumptions to consider, namely as far as withdrawal, retirement, disability or mortality rates are concerned; and an economic basis, which deals with the economic assumptions underlying, in general terms, interest rates. The *IAN* (Perryman, 1949) is used. Let i denote the annual effective interest rate, $v = \frac{1}{1+i}$ denote the associate discount factor, both used to evaluate payments at specific dates, and ${}_tp_x$ denote the probability that a life aged x survives for t years.

Contingent on life status, life annuities depend on the following *rvs*:

- the future lifetime *rv*, denoted by T_x, which represents the lifetime of a person aged x, in years;
- the curtate future lifetime *rv*, denoted by $K_x = \lfloor T_x \rfloor$, which represents the lifetime of a person aged x, in complete years;
- and the m-thly curtate future lifetime *rv*, denoted by $K_x^{(m)} = \frac{\lfloor m \cdot T_x \rfloor}{m}$, which represents the lifetime of a person aged x, in complete years, measured through m parts of a year.

Life annuities are nowadays a very important piece of every day’s life, even if people don't recognize their importance. Salaries, pensions, bonds, etc. are all examples of every day’s tools which constitute or may be valued using annuities. Furthermore, life annuities are known to provide a steady income through a specified term, be it under retirement, sickness or disability, constituting a safe mean of sustaining survival and guarantying protection against longevity and financial risk.

2.1.1 Traditional life annuities

2.1.1.1 Whole life annuities

Whole life annuities are life annuities whose payments are contingent on a life being alive at each payment date, ceasing when the life dies. This is the most common type of life annuity. For instance, the social security pension is a whole life annuity.

In the annual case, payments are made each year provided a life is alive at each payment date. In the $1/m$-thly case payments are made $1/m$-thly throughout the year, provided the life is alive at the payment date, where m is the number of equal periods of time the year is divided into. For instance, if $m = 12$, then the year is divided into 12 months and the payments are made each month. In the continuous case, the payments are made continuously, provided the life is alive. In addition to these distinctions, annuities may be paid in advance or in arrears, i. e., payments may be made at the beginning or at the end of each period, with notation applying, respectively, as $\ddot{a}$ and a. Of course, this distinction only makes sense in the annual and $1/m$-thly cases.

Table 1 summarizes the *PV rvs* and the correspondent *EPVs* for each of these whole life annuity contracts.

Case	Annual		1/*m*-thly		Continuous
	In Advance	In Arrears	In Advance	In Arrears	
PV rv	$\ddot{a}_{\overline{K_x+1}\rvert}$	$a_{\overline{K_x}\rvert}$	$\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}\rvert}$	$a^{(m)}_{\overline{K_x^{(m)}}\rvert}$	$\bar{a}_{\overline{T_x}\rvert}$
EPV	$\ddot{a}_x = \sum_{t=0}^{\infty} v^t \, {}_tp_x$	$a_x = \ddot{a}_x - 1$	$\ddot{a}_x^{(m)} = \sum_{t=0}^{\infty} \frac{1}{m} v^{\frac{t}{m}} \, {}_{\frac{t}{m}}p_x$	$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$	$\bar{a}_x = \int_0^{\infty} e^{-t} \, {}_tp_x \delta t$

Table 1 - Whole Life Annuities

For all the following types of life annuities, the three cases, annual, $1/m$-thly and continuous, as well as the distinctions between payments in advance or in arrears, hold.

The whole life annuities described above are level (all payments of the same amount) and pay 1 per year. Unless otherwise stated, annuities are assumed to be level with payments of 1 per year.

2.1.1.2 Term annuities

Term annuities are life annuities whose payments are contingent on a life being alive at each payment date, during a specific period of time, a term, ceasing when the life dies within the term or if the life has survived the term (in which case all payments are made) whichever occurs first. This is also a very common type of life annuity. For instance, children's pension in case of a parent's death, until the child reaches the adult age is the most common case. Usually the term is a period of n years, but it may be quarters, months or even days.

Table 2 summarizes the *PV rvs* and the correspondent *EPVs* for each of the term life annuity contracts.

Case	Annual		1/*m*-thly		Continuous
	In Advance	In Arrears	In Advance	In Arrears	
PV rv	$\ddot{a}_{\overline{min(K_x+1,n)}\vert}$	$a_{\overline{min(K_x,n)}\vert}$	$\ddot{a}^{(m)}_{\overline{min\left(K_x^{(m)}+\frac{1}{m},n\right)}\vert}$	$a^{(m)}_{\overline{min\left(K_x^{(m)}+\frac{1}{m},n\right)}\vert}$	$\bar{a}_{\overline{min(T_x,n)}\vert}$
EPV	$\ddot{a}_{x:\overline{n}\vert} = \sum_{t=0}^{n-1} v^t\,{}_tp_x$	$a_{x:\overline{n}\vert} = = \ddot{a}_{x:\overline{n}\vert} - -1 + v^n\,{}_np_x$	$\ddot{a}^{(m)}_{x:\overline{n}\vert} = = \sum_{t=0}^{n\cdot m-1} \frac{1}{m} v^{\frac{t}{m}}\,{}_{\frac{t}{m}}p_x$	$a^{(m)}_{x:\overline{n}\vert} = \ddot{a}^{(m)}_{x:\overline{n}\vert} - -\frac{1}{m}(1 - v^n\,{}_np_x)$	$\bar{a}_{x:\overline{n}\vert} = = \int_0^n e^{-t}\,{}_tp_x \delta t$

Table 2 - Term Annuities

Basically whole life and term annuities, either level or variable, are the foundation of life annuities. The following types of life annuities simply add up features to these basic definitions, such as a deferral or a guaranteed period, as a way to mitigate for the rigidity of life annuities and to serve policyholders needs.

2.1.1.3 Deferred life annuities

Deferred life annuities are life annuities whose payments are deferred for a period of time before the first payment date occurs, called the deferral period. The deferral period is usually a n-year period, but of course, it might be a period of n quarters, months, days or any other relevant time interval. From the first payment on, the annuity may behave as a whole life annuity or as a term annuity according to the definitions set above. This type of life annuity is becoming more and more popular since mortality improvements are affecting retirement dates and death is farther away. A life annuity is quite expensive and so people might want to defer paying such a high price for something that is in its nature definitive.

The main relation between the *EPV* of deferred whole life annuities and those of immediate whole life and term annuities is summarized below:

$$_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n\,{}_np_x\ddot{a}_{x+n}.$$

2.1.1.4 Guaranteed life annuities

Under a guaranteed life annuity, payments are guaranteed (certain) for a period of time, the guaranteed period, before the first contingent payment occurs. Similarly to the deferral period, the guaranteed period is usually a n-year period, but of course, it might be a period of n quarters, months, days or any other relevant time interval. From the end of the guaranteed period on, the annuity may behave as a whole life annuity or as a term annuity. This is a particularly useful feature when referring to retirees who leave an elderly spouse. To support the higher costs after death or adjusting to a single life might take longer than expected and thus having a steady income stream is helpful.

The main connection between the *EPV* of guaranteed life annuities and those of whole life and term annuities is summarized below:

$$\ddot{a}_{\overline{x:\overline{n|}}} = \ddot{a}_{\overline{n|}} + {}_{n|}\ddot{a}_x = \ddot{a}_{\overline{n|}} + v^n {}_n p_x \ddot{a}_{x+n}.$$

2.1.1.5 Geometrically increasing life annuities and inflation protection

Geometrically increasing life annuities are life annuities whose payments are geometrically increasing, i. e., payments increase at a rate $j > 0$ per year. This type of life annuities is very popular, since they are a good mean to offset the effects of inflation on the standard of living and on the income received to support it. When this purpose is explicitly declared, they are termed inflation-protected life annuities.

The *EPVs* for this type of life annuities are basically equal to the ones for whole life and term annuities, but simply computed at the transformed annual effective rate $i^* = \frac{i-j}{1+j}$.

2.1.1.6 Joint and last survivor life annuities

The life annuities presented are contingent on only one life. Nevertheless, life annuities may be contingent on more than one life. The simplest and most common case is the one which considers contingency on two lives, who may be of the same gender. The distinction between genders is important because mortality applying to men and women is usually not the same. As experience has shown, women tend to have a longer life span than men and, though it is now illegal across Europe to compute premiums based on gender, still different mortality assumptions apply.

The following life annuities are contingent on two lives. The possible cases are extended when comparing with the one life cases, though the notation is similar, with both lives' ages in subscript. For the purpose of illustration only, the continuous case applies.

Joint life annuities are life annuities whose payments are contingent on two lives (x) and (y) (respectively aged x and y), payable provided both lives are alive at each payment date. The annuity may behave as a whole life annuity or a term annuity, according to the definitions set above, ceasing at the first death. The *EPV* for the continuous case is denoted by $\bar{a}_{xy} = \int_0^{\infty} e^{-t} \, {}_tp_{xy} \delta t$, where ${}_tp_{xy}$ is the probability that both lives (x) and (y) are alive at time t.

Last survivor life annuities are life annuities whose payments are contingent on two lives (x) and (y), payable as long as at least one of the lives is alive at each payment date. The annuity may be whole life or term, ceasing at the last death. This type of life annuity is usually found in life insurance contracts for a couple who wishes to receive a certain income even if one of the members dies. The *EPV* for the continuous case is given by $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$.

2.1.1.7 Reversionary life annuities

The simplest example of a reversionary life annuity is that of an annuity whose payments are contingent on two lives (x) and (y), payable provided life (x) has already died and life (y) is alive at each payment date. The annuity may be whole life or term, commencing at (x)'s death and ceasing at (y)'s death. Usually, (x) and (y) represent a married couple, in which case the wife receives a pension for life upon the husband's death. The *EPV* for the continuous case is given by $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$.

To evaluate life annuities some approximations must be used since formulae become very complicated and computationally heavy. The broadly accepted approximation methods are the *UDD* Assumption and Woolhouse's Formula. More specific insight into these topics may be gathered from Dickson *et al.* (2009).

2.1.2 More recent life annuities

More recently exotic types of annuities have been developed, to account for the increase in demand and competition of providers. Just a few examples include commutable, equity-indexed and variable life annuities. Commutable life annuities, already mentioned when introducing Wang and Young (2011), are a more flexible type of life annuities in the way that they hold a surrender option. In other words, commutable life annuities are life annuities which behave as whole or term life annuities, except for the fact that there is a surrender option allowing the annuitant to borrow against or surrender any proportion of the annuity at any time as long as the life is still alive. The surrender value is set to be a proportion of the purchase value, paid at surrender.

Equity-indexed life annuities (*EIAs*), cf. Bauer *et al.* (2006) for a detailed survey, are linked to stock market performance in such a way that, when a stock index goes up, *EIAs* provide annuitants with a rate of return in line with the return of the index and, when the stock index goes down, *EIAs* provide annuitants with a minimum guaranteed rate of return. The so-called 'variable annuities' are life annuities which offer a range of investment options. The value of a variable annuity varies depending on the performance of the investment options chosen, which are typically linked to mutual funds that invest in stocks, bonds, money market instruments or some combination of these, offering death benefits to pre specified beneficiaries and tax deferment, i. e., income and investment are only subject to taxes upon withdrawal.

2.2 Basics on Ruin Theory

Most of the following concepts may be found in introductory books on Risk Theory, such as Dickson (2005) or Grandell (1991).

Ruin Theory, as a subject under the Risk Theory analysis, means to study the wealth of portfolios, companies or even individuals over time as to assess the probable occurrence of financial ruin. The figure which is widely accepted to best describe wealth is the Reserve or Net Wealth, as described by Milevsky and Robinson (2000).

To model reserves over time all relevant factors affecting cash flows should be considered: claim payments, premiums, investment income and expenses are the most important. Since there is no certainty regarding the number of claims and the total amount of claims at the beginning of any considerable period there is the need to determine the reserve required to provide for those liabilities. The objective of Ruin Theory's output is not to obtain a thorough representation of reality but rather to provide information on the risks associated with portfolio management, which is a considerably useful tool for long-run financial planning.

In the Ruin Theory framework, several variables are constantly used with the changes occurring in the way these variables are modeled. The general convention is to consider reserves as either discrete or continuous time processes, denoted by $U(t)$, with $t \in \mathbb{N}_0$ or $t \geq 0$, accordingly, and $U(t) \in \mathbb{R}$. If the initial reserve is denoted by $U(0) = u$, then the general expression for the reserve at time t may be $U(t) = u + P(t) + C(t) - S(t)$, where $P(t)$ is the premium process, which measures premiums (written or earned) collected up to time t, net of expenses, $S(t)$ is the claims process, which measures claims incurred or paid up to time t and $C(t)$ represents any cash-flow other than the collection of premiums or the payment of claims at time t, such as investment income or expenses. Expenses and other cash flows are usually irregular and more difficult to measure, so no further analysis will be carried out regarding these processes.

Usually the premium process is the easiest to model, even though it may be correlated with either $S(t)$ or $C(t)$. The most common definition for the collection of premiums is to assume a fixed amount per period of time, i. e., assume $P(t) = ct$, where $c > 0$ is called the premium rate. However, premiums may be defined in different ways considering the type of business. When considering portfolios of life annuities, the premium is either paid only once, at inception, if the annuity is immediate or regularly paid, during the deferment period, if the annuity is deferred.

2.2.1 Collective and Individual Risk Models

As it is easily understood, there are several ways to model the claims probabilistic distribution, depending on the type of business and the perspective adopted. The approaches are immensely varied but it has been pointed out in Chapter 1 that a main distinction in literature is to consider collective and individual risk models. These models may then be applied to general insurance, life insurance and particular lines of business accordingly.

The claims under a life annuity portfolio are dependent on the survival of the annuitant to each payment date. If the annuitant is alive at the payment date, then the payment has to be made and this is a claim. The number of claims then depends on mortality assumptions and the amount of each claim depends on the benefit definition. A collective risk model (*CRM*) considers the number and severity of claims arising from a portfolio as a whole, whilst an individual risk model (*IRM*) considers the number and severity of claims arising from the individual policies or accounts in the portfolio. The *CRM* has been the most developed and widely used risk model considering the simplicity of conclusions and the natural application to general insurance.

The *IRM* assumes a portfolio of n independent policies, for which, in each period, either one claim occurs, with probability p_i, or no claim occurs, with probability $q_i = 1 - p_i$,

$i = 1,2, \dots, n$. This type of model framework makes sense when applied to life insurance. This is because considering only one claim in each period is consistent with the idea of assuming dependency on life status, i.e., a claim may occur on death, disability, sickness or survival and such occurrences are mutually exclusive. For this reason, the probability q_i is usually a death rate following a pre-determined suitable mortality model. The idea is to determine the total aggregate claim amount to be paid to the policies in the portfolio, denoted by $S(t)$. Of course,

$$S(t) = \sum_{i=1}^{n} S_i(t), \tag{1}$$

where $S_i(t)$ is the total individual claim amount per policy i at time t, depending on the claim amount to be paid $b_i(t)$, which may be either deterministic or random. Obviously,

$$E[S(t)] = \sum_{i=1}^{n} E[S_i(t)]$$

and, under the assumption that policies are independent,

$$Var\big(S(t)\big) = \sum_{i=1}^{n} Var\big(S_i(t)\big).$$

In general,

$$E[(S(t) - E[S(t)])^k] = \sum_{i=1}^{n} E[(S_i(t) - E[S_i(t)])^k]. \tag{2}$$

When the claim amount is not fixed, the probability function of $b_i(t)$ may be denoted $f_{b_i(t)}(b) = P(b_i(t) = b)$.

This is the fundamental IRM framework, to be used in the following of the present work. The primary goal is to determine the distribution of $S(t)$, which essentially may be determined by convoluting the distributions of $\{S_i(t)\}_{i \in \{1,2,\dots,n\}}$.

Following the definitions by Gerber (1997), the distributions of $S_1(t) + S_2(t)$, $S_1(t) + S_2(t) + S_3(t)$, …, $S_1(t) + S_2(t) + S_3(t) + \cdots + S_n(t)$ are successively computed, in such a way that the distribution of $S_1(t) + S_2(t) + \cdots + S_i(t)$ may be determined using the formula:

$$P(S_1(t) + S_2(t) + \cdots + S_i(t) = s) =$$

$$= \sum_{j=1}^{m} P\big(S_1(t) + S_2(t) + \cdots + S_{i-1}(t) = s - s_{ij}\big) q_i + P(S_1(t) + S_2(t) + \cdots + S_{i-1}(t) = s) p_{ij} \tag{3}$$

where $\{s_{ij}\}_{i\in\{1,2,\dots,n\};j\in\{1,2,\dots,m\}}$ represents the possible values for $S_i(t)$, $q_i = P(S_i(t) = 0)$ and $p_{ij} = P\big(S_i(t) = s_{ij}\big)$. This approach involves a great deal of time consumption and so, over the years, several authors have looked for alternative methods.

2.2.2 The De Pril Recursion Formulae

The contribution from De Pril to the evaluation of the distribution of $S(t)$ under the *IRM* was performed in several stages. De Pril (1986) determined an exact recursion formula for the distribution of $S(t)$, considering the amount of each claim to be fixed. The author generalized later the result (De Pril 1989) to "positive arbitrary claim amounts", to include the possibility that the amount of each claim is not fixed, but a *rv*.

In De Pril (1989), the author determined two exact recursion formulas to compute the *pdf* of $S(t)$: one using convolutions of $f_{b_i(t)}(b)$, which may be determined using (3), and another one, described next. In both works, De Pril provides related approximations to mitigate for the long computation time of the exact procedures using convolutions.

For the De Pril (1989) recursion to be applied, the portfolio has to be divided by the life tables applicable and by the possible benefit amounts, as to create homogeneous 'sub portfolios' with respect to mortality and benefits. Each of the n individuals is subject to mortality following one of L possible life tables and entitled to one of B integer benefit amounts, in such a way that $n = \sum_{b=1}^{B}\sum_{l=1}^{L} n_{l,b}$, where $n_{l,b}$ represents the number of annuitants subject to mortality following the life table l and entitled to a benefit of amount b. The choice of life table may depend on several factors, namely gender, region, occupation or age range. In addition, the benefit amounts, assumed $b = 1,2,\dots,B$ for the purposes of applying this model, are integral multiples of an appropriate monetary unit.

Let $f_{S(t)}(s) = P(S(t) = s)$ and $F_{S(t)}(s) = P(S(t) \le s) = \sum_{k=1}^{s} f_{S(t)}(k)$ represent the *pdf* and the *cdf* of $S(t)$, respectively. The De Pril recursion is a two stage approach in such a way that the probability of a null total aggregate claim amount, i. e., the probability that there are no claims in a period is given by

$$f_{S(t)}(0) = \prod_{b=1}^{B}\prod_{l=1}^{L}(q_l)^{n_{l,b}}, \tag{4}$$

and the probability that the total aggregate claim amount in a period is equal to S^* is given by

$$f_{S(t)}(S^*) = \frac{1}{s}\sum_{b=1}^{\mathrm{B}}\sum_{l=1}^{L} n_{l,b}\sum_{s=1}^{S^*} w_{l,b}(s) f_{S(t)}(S^* - s), \tag{5}$$

with

$$w_{l,b}(s) = \frac{{}_tp_l}{{}_tq_l}\left(s f_{b_i(t)}(r) - \sum_{r=1}^{s} f_{b_i(t)}(r)\, w_{l,b}(s-r)\right), \tag{6}$$

for $s \in \mathbb{N}$ and $w_{l,b}(0) = 0$, representing a factor of adjustment in the formula (see De Pril (1989), pages 11-12).

These recursions have driven a considerable amount of interest since their appearance and a number of models, mostly in the form of approximations, followed.

2.2.3 Alternative Models

2.2.3.1 The Dhaene & Vandebroek model

Dhaene and Vandebroek (1995) compared the exact calculation derived by De Pril with their own exact calculation, only to find that their recursion formula performs well for less heterogeneous portfolios, whereas De Pril is preferred for more heterogeneous ones. The method is determined by:

$$f_{S(t)}(0) = \prod_{b=1}^{B}\prod_{l=1}^{L}(q_l)^{n_{l,b}}$$

and

$$f_{S(t)}(S^*) = \frac{1}{S^*}\sum_{b=1}^{\mathrm{B}}\sum_{l=1}^{L} n_{l,b} w_{l,b}(S^*),$$

with

$$w_{l,b}(s) = \frac{{}_tp_l}{{}_tq_l}\left(\sum_{r=1}^{s} f_{b_i(t)}(r)\left(r f_{S(t)}(s-r) - w_{l,b}(s-r)\right)\right),$$

for $s \in \mathbb{N}$ and $w_{l,b}(0) = 0$, representing a factor of adjustment in the formula (see Dhaene and Vandebroek (1995), pages 11-12).

2.2.3.2 The Kornya approximation

The Kornya (1983) approximation to compute the distribution of $S(t)$ under the *IRM* lies under the same cornerstones as the De Pril method, but works as an approximation rather than as an exact calculation. The initial condition is exactly the same.

This model was generalized to the non-fixed claim amount case, as in De Pril (1989), by Hipp (1986), with the particular property that the first order approximation coincides with the usual compound Poisson approximation (see next paragraphs) in the *CRM*. De Pril (1986, 1989) also determined smaller error bounds for the Kornya and Hipp's approximations.

2.2.3.3 The compound Poisson approximation

This represents the most classical form of approximation to the distribution of $S(t)$, especially if the collective model is to be adopted. This approach (Dickson, 2005) makes use of the fact that the simple compound Binomial distribution of $S_k(t)$, for $k = 1,2, \dots, n_{lb}$, representing the *cdf* for each group within the portfolio, may be approximated by a compound Poisson distribution and so $S(t)$ may be approximated by a compound Poisson process, as well. For this approximation to be sufficiently good, the n_{lb} must be high, the q_l must be low and the product $n_{lb}q_l$ must be constant.

The approximation suggested by De Pril (1989) also gives smaller error bounds than the compound Poisson approximations and should be preferred.

2.2.3.4 The Normal approximation

This is a relatively simple approach to take. The idea is to assume $S(t)$ as a Normal distributed *rv* (Pentikainen 1987) with parameters $\mu = E[S(t)]$ and $\sigma^2 = Var\big(S(t)\big)$, in such a way that, following the *CLT*, $S^*(t) = \frac{S(t)-\mu}{\sigma} \sim N(0,1)$. Let $\Phi(\cdot)$ be the *cdf* of a Standard Normal *rv*. Then, the *cdf* of $S(t)$ is in this case approximated by:

$$F_{S(t)}(s) \approx \Phi\big(S^*(t)\big). \tag{7}$$

Though using the Normal is convenient for its simplicity, it is often not realistic. A normality behavior, characterized by symmetry around the mean, is not a feature possessed by many of the variables of interest. As a result, the Normal approximation is only appropriate if the portfolio's size is sufficiently large and relatively homogeneous, to create a compensation between the non-normal effects. Usually, in the case of insurance, claims are skewed (Ramsay 1991) and so the tails of the distribution are harder to fit using a Normal approximation.

Alternatives to the Normal approximation may be the use of the Normal Power (*NP*) or the Translated Gamma approximations.

2.2.3.5 The NP approximation

This approximation follows the same approach of using a standardized normal distribution, but furthermore takes into account the possible asymmetric form of $S(t)$ by considering a function $S^N(t) = \nu\left(S^*(t)\right)$, with $S^N(t) \sim N(0,1)$, which transforms the standardized $S(t)$ into a symmetric or quasi-symmetric *rv* well enough approximated by a standardized normal (Pentikainen 1987). The *NP* Approximation is based on the Edgeworth's Series applied to the *cdf* of $S^*(t)$ (Beard *et al.*, 1984) and assumes that $S^*(t) = \nu^{-1}\left(S^N(t)\right)$ follows a quadratic function of the form:

$$S^*(t) \approx aS^N(t) + b\left(\left(S^N(t)\right)^2 - 1\right). \tag{8}$$

The standard *NP* approximation assumes $a = 1$ and $b = \frac{\gamma_S}{6}$, where γ_S is the skewness coefficient of $S(t)$. But it is only possible to proceed with this approach if $0 < \gamma_S < 1$ and the approximation is as good as γ_S is not close to these limiting values. In this case, the *cdf* of $S(t)$ is approximated by:

$$F_{S(t)}(s) \approx \Phi\left(-\frac{3}{\gamma_S} + \sqrt{\frac{9}{\gamma_S^2} + 1 + \frac{6}{\gamma_S}\frac{s - \mu_S}{\sigma_S}}\right), \tag{9}$$

where μ_S and σ_S are, respectively, the mean and standard deviation of $S(t)$.

This approach is restrictive since many variables may be negatively skewed instead or positively skewed at a higher rate. To account for a broader range of γ_S values, the adjusted *NP* approximation (Ramsay, 1991) may be used. In this case, a further degree of matching is introduced: the method of moments is applied to match the first three central moments in (8). In this way, the parameters a and b are obtained as the solutions of the following system of equations:

$$\begin{cases} 1 = a^2 + 2b^2 \\ \gamma_S = 6a^2b + 8b^3 \end{cases} <=> \begin{cases} a^2 = 1 - 2b^2 \\ \gamma_S = 6b - 4b^3 \end{cases}.$$

This approach is good for $-2\sqrt{2} \leq \gamma_S \leq 2\sqrt{2}$, so that the system has one single root in $-\frac{1}{\sqrt{2}} \leq b \leq \frac{1}{\sqrt{2}}$. The *cdf* of $S(t)$ is approximated by:

$$F_{S(t)}(s) \approx \Phi\left(-\frac{a}{2b} + \sqrt{\frac{a^2}{4b^2} + 1 + \frac{1}{b}\frac{s - \mu_S}{\sigma_S}}\right). \tag{10}$$

2.2.3.6 The translated Gamma approximation

Using the translated Gamma approximation (Centeno, 2003) is a similar approach as to transform $S(t)$ into a Gamma distributed *rv* added by a constant. This is done by guaranteeing that $S(t)$ and the new *rv*, say $S^G(t) = g + G(t)$, where g is a constant and $G(t) \sim Gamma(\alpha, \theta)$, have the same mean, variance and skewness coefficient. The parameters s, α and θ are obtained in this way and the *cdf* of $S(t)$ is approximated by:

$$F_{S(t)}(s) \approx P(g + G(t) \leq s) = 1 - P(G(t) \leq s - g). \tag{11}$$

The results from the translated Gamma approximation are very similar to those of the *NP* approximation.

2.2.4 Probabilities of Survival and Ruin

To close this chapter, ruin and survival probabilities are introduced (Klugman *et al.*, 2008). Following the generally accepted notation, the ruin probability and its complementary, the survival probability, from an initial reserve level u, are denoted, respectively, by $\Psi(\cdot)$ and $\Phi(\cdot) = 1 - \Psi(\cdot)$ and are defined in the following four ways, differing according to two time categories: discrete/continuous and finite/infinite.

1) Continuous-Time, Infinite-Horizon - this definition assumes that the amount of reserve is continuously checked and the portfolio is, under survival, infinitely solvent:

$$\Phi(u) = P(U(t) \geq 0, \forall\, t \geq 0 | U(0) = u)$$

and so

$$\Psi(u) = P(\exists\, t \geq 0 \colon U(t) < 0 | U(0) = u). \tag{12}$$

2) Discrete-Time, Infinite-Horizon - the amount of surplus is checked usually at the end of discrete periods of time, such as days, months, quarters or years, and the portfolio is, under survival, infinitely solvent:

$$\widetilde{\Phi}(u) = P(U(t) \geq 0, \forall\, t \in \mathbb{N}_0 | U(0) = u)$$

and so

$$\widetilde{\Psi}(u) = P(\exists\, t \in \mathbb{N}_0 : U(t) < 0 | U(0) = u). \tag{13}$$

3) Continuous-Time, Finite-Horizon - the amount of surplus is continuously checked, but the portfolio is, under survival, finitely solvent, i. e., is solvent over τ periods of time (days, months, quarters, years):

$$\Phi(u, \tau) = P(U(t) \geq 0, \forall\, t \in [0, \tau] | U(0) = u)$$

and so

$$\Psi(u, \tau) = P(\exists\, t \in [0, \tau] : U(t) < 0 | U(0) = u). \tag{14}$$

4) Discrete-Time, Finite-Horizon - the amount of surplus is discretely checked and the portfolio is, under survival, solvent until τ:

$$\widetilde{\Phi}(u, t) = P(U(t) \geq 0, \forall\, t \in \{0, 1, \dots, \tau\} | U(0) = u)$$

and so

$$\widetilde{\Psi}(u, \tau) = P(\exists\, t \in \{0, 1, \dots, \tau\} : U(t) < 0 | U(0) = u). \tag{15}$$

The main relationship between these probabilities is the fact that, from finite to infinite time horizon, convergence is swift enough for the matter of choosing amongst the various definitions to be set upon the ease of calculation and the appropriateness of the model, rather than on the impossibility of studying infinite time horizon probabilities,

$$\lim_{\tau \to \infty} \Phi(u, \tau) = \Phi(u) \text{ and } \lim_{\tau \to \infty} \widetilde{\Phi}(u, \tau) = \widetilde{\Phi}(u).$$

Chapters 3 and 4 are devoted to a case study. In Chapter 3 the theoretical framework is developed and in Chapter 4 the solution for the problem in question is searched.

Chapter 3 – Case Study, the Theory

3.1 The Problem

Consider a real life insurance company that (to be consistent with the previous notation) holds a particular portfolio of n whole life annuities. The annuitants are aged between x_{min} and x_{Max} and L different mortality models are at use. There are B possible benefit amounts, payable *pa*.

As stated at the end of Chapter 1, the idea is to follow the work by Frostig and Denuit (2009) and apply the classic individual risk model to the real data gathered using the guidelines given in their paper.

The problem to be solved is a double sided problem: (1) to determine the probability that the insurer will not be able to provide for all the payments due to all annuitants until their deaths, given the initial reserve u; (2) to allocate u over the homogeneous in order to maximize the joint survival probability.

A more formal description of the problem is presented at the end of the chapter, after all the required notation and terminology have been introduced.

3.2 The Model

Frostig and Denuit (2009) evaluated ruin probabilities for a closed portfolio of n heterogeneous annuitants. The authors divided the portfolio into M homogeneous classes, to be able to apply an *IRM*, such as De Pril (1989).

Each class is characterized according to the age of the annuitant, the applicable life table, and the benefit, as described in 2.2.2, so that in class m each annuitant is aged x_m, with mortality following life table l_m and a benefit amount b_m. In this way, $T_{m,i}$ represents the future lifetime, in years, of the i^{th} life in the m^{th} group, with n_m representing the total number of annuitants in group m and $n_1 + n_2 + \cdots + n_M = n$.

For a portfolio of whole life annuities, a crucial aspect is the timeframe of payments, which refers to the difference between the limiting age ω (survival beyond age ω is not possible) and the minimum age in the portfolio x_{min}. Let $\tau = \omega - x_{min}$ denote this difference, with $\tau \in \mathbb{N}$.

As seen in Chapter 2, the initial reserve $U(\tau)$ required to support the portfolio until time τ is a *rv* equal to the *PV* of all the future payments made to the n annuitants,

$$U(\tau) = \sum_{m=1}^{M} \sum_{i=1}^{n_m} b_m a_{\overline{\min\{T_{m,i};\tau\}}|} = \sum_{m=1}^{M} \sum_{i=1}^{n_m} b_m \sum_{t=1}^{\tau} (1+r)^{-t} I(T_{m,i} > t), \tag{16}$$

where r represents the constant term structure of interest rates for the period $[0;\tau]$ and $I(T_{m,i} > t) = \begin{cases} 1, & \text{if } T_{m,i} > t \\ 0, & \text{if } T_{m,i} \leq t \end{cases}$ is an indicator function depending on the life status of the i^{th} annuitant. The actual initial reserve is the price paid by all the annuitants in the portfolio at inception, denoted $U(0) = u$. For each of the m groups, the initial reserve is denoted $u_1, u_2, \ldots, u_m$, such that $u = \sum_{m=1}^{M} u_m$.

In the same way, it is possible to define the *PV rv* of all the payments made to the i^{th} annuitant in group m, up to time τ,

$$U_{m,i}(\tau) = b_m \sum_{t=1}^{\tau} (1+r)^{-t} I(T_{m,i} > t), \quad i = 1, 2, \ldots, n_m. \tag{17}$$

Furthermore,

$$U(\tau) = \sum_{m=1}^{M} U_m(\tau) = \sum_{i=1}^{M} \sum_{i=1}^{n_m} U_{m,i}(\tau). \tag{18}$$

Under the assumption that the *rvs* $U_{m,i}(\tau)$ are mutually independent and identically distributed, the authors stated that the distribution of $U(\tau)$ might be computed using the recursive formulae derived for the *IRM* by De Pril (1989). In this case,

$$f_{U(\tau)}(0) = \prod_{m=1}^{M} (q_{x_m})^{n_m}, \tag{19}$$

is the probability that all annuitants die in the first year and so there are no payments to be made; further,

$$f_{U(\tau)}(U) = \frac{1}{U}\sum_{m=1}^{M} n_m \sum_{u=1}^{U} w_m(u) f_{U(\tau)}(U-u), \tag{20}$$

is the probability that the aggregate claim amount in a given period is equal to U,

$$w_m(u) = \frac{{}_\tau p_{x_m}}{{}_\tau q_{x_m}}\left(u f_{b_i(t)}(u) - \sum_{y=1}^{u} f_{b_i(t)}(y)\, w_m(u-y)\right), \tag{21}$$

$u, y \in \mathbb{N}$, $w_m(0) = 0$, and $f_{b_i(\tau)}(b) = f_m(b) = P\left(b_m a_{\overline{\min\{T_{m,\iota};\tau\}}|} = b\right), b = 1,2,\dots,B,\ \ m = 1,2,\dots,M$. This is an extension of the De Pril recursion formula, since there are $M = B \times L$ benefit distributions, considering together the benefit amounts and the mortality rates, instead of only B.

To apply this formula to real data the possible benefit amounts to be paid must be multiples of a pre-determined monetary unit (*mu*), such that $b = 1,2,\dots,B$ is their full representation. For this matter, the Dispersion Method developed by Gerber (1997), which is a quite simple method to apply, may be considered. The idea is to set a monetary unit MU, such that each of the original b_m amounts become $b_m^* = \frac{b_m}{MU}$. Then these values are to be multiplied by all the possible values for $a_{\overline{\min\{T_{m,\iota};\tau\}}|}$, considering discounting factors only, and dispersed through the integers using the associated probabilities of occurrence.

However, there is a note to consider when applying the De Pril recursion formula to life annuities' portfolios, which concerns the initial probability, $f_{U(\tau)}(0)$. In the present case, the initial condition is dependent on the death of all annuitants in the first period, which is obviously a very improbable event. Even for older ages, for instance above 80's, the initial probability would become equal or very close to 0 (*cf.* (4) and (5)) and so the recursion wouldn't be able to come out of a null result. Nevertheless, it is possible to follow an alternative approach to surpass this problem: using a fictitious non-null initial probability, as described in Klugman *et al.* (2008, page 230). Such approach would work as follows: 1) an initial probability, distinguishable from 0, would be set, say $f_{U(\tau)}(0) = 20\%$; 2) the recursion formula would be applied using this starting point; 3) the whole range of probabilities obtained would finally be summed and adjusted accordingly, to guarantee a real result, using

$f^*_{U(\tau)}(t) = \frac{f_{U(\tau)}(t)}{\sum_{t=1}^{\tau} f_{U(\tau)}(t)}$. Such approach presented itself later in the development of the work and for this reason was not concluded.

Perhaps, this difficulty in the application of De Pril's recursion is also the reason behind the fact that Frostig and Denuit (2009) ended up using only moment-based approximations to the *cdf* of $U(\tau)$. For the Normal approximation it is necessary to determine the mean and standard deviation of $U(\tau)$. For the *NP* approximation, there is the additional need to find the skewness coefficient.

3.2.1 Auxiliary results

The relevant probabilities are $p_{x_m+t} = P\big(T_{m,i} > t+1 \,\big|\, T_{m,i} > t\big) = \frac{{}_{t+1}p_{x_m}}{{}_{t}p_{x_m}}$ and ${}_{t}p_{x_m} = \prod_{s=0}^{t-1} p_{x_m+s}$, adopting the *IAN*. The indicator function $I\big(T_{m,i} > t\big)$ is itself a *rv*, of which the moments of $U(\tau)$ are dependent. So it is important to determine the following raw moments:

$$E\left[\left(I\big(T_{m,i} > t\big)\right)^k\right] = 1^k P\big(T_{m,i} > t\big) + 0^k P\big(T_{m,i} \le t\big) = P\big(T_{m,i} > t\big) = {}_{t}p_{x_m}$$

$$\begin{aligned} E\big[I\big(T_{m,i} > t_1\big) \ldots I\big(T_{m,i} > t_k\big)\big] &= 1P\big(T_{m,i} > t, \forall\, t\ \in \{t_1, \ldots, t_k\}\big) + 0P\big(\exists\, t\ \in \{t_1, \ldots, t_k\}: T_{m,i} \le t\big) = \\ &= P\big(T_{m,i} > Max(t_1, \ldots, t_k)\big) = {}_{Max(t_1, \ldots, t_k)}p_{x_m} \end{aligned}$$

From these results it is then possible to determine the mean and variance of $U(\tau)$, as well as the skewness coefficient (Frosting and Denuit (2009) only presented the mean and variance, to apply the Normal approximation).

The mean of $U_{m,i}(\tau)$ is given by

$$\begin{aligned} \mu_{m,i}(\tau) = E[U(\tau)] = E\left[b_m \sum_{t=1}^{\tau} (1+r)^{-t} \cdot I\big(T_{m,i} > t\big)\right] = \\ = b_m \sum_{t=1}^{\tau} (1+r)^{-t} E\big[I\big(T_{m,i} > t\big)\big] = b_m \sum_{t=1}^{\tau} (1+r)^{-t}\, {}_{t}p_{x_m} = \\ = b_m \sum_{t=1}^{\tau} (1+r)^{-t} \prod_{s=0}^{t-1} p_{x_m+s}. \end{aligned} \tag{22}$$

Then, the mean of $U_m(\tau)$,

$$\mu_m(\tau) = E[U_m(\tau)] = E\left[\sum_{i=1}^{n_m} U_{m,i}(\tau)\right] = \sum_{i=1}^{n_m} \mu_{m,i}(\tau) \tag{23}$$

and, finally, the mean of $U(\tau)$ is,

$$\mu(\tau) = E[U(\tau)] = E\left[\sum_{m=1}^{M} U_m(\tau)\right] = \sum_{m=1}^{M} \mu_m(\tau). \tag{24}$$

The variance may be computed from two different approaches:

1) using the variances and covariances of $I(T_{m,i} > t)$;

2) using $Var\left(U_{m,i}(\tau)\right) = E\left[\left(U_{m,i}(\tau)\right)^2\right] - \left(E\left[U_{m,i}(\tau)\right]\right)^2$.

For the purpose of Excel computations the second approach is preferable, and a useful relationship will be derived next. The second raw moment of $U_{m,i}(\tau)$ is given by

$$\begin{aligned}
E\left[\left(U_{m,i}(\tau)\right)^2\right] &= E\left[\left(b_m \sum_{t=1}^{\tau} (1+r)^{-t} I(T_{m,i} > t)\right)^2\right] = \\
&= b_m^2 E\left[\sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-t} I(T_{m,i} > t)(1+r)^{-k} I(T_{m,i} > k)\right] = \\
&= b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} E\left[I(T_{m,i} > t) I(T_{m,i} > k)\right] = \\
&= b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} {}_{Max(t,k)}p_{x_m} = \\
&= b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} \prod_{s=0}^{Max\{k-1;t-1\}} p_{x_m+s},
\end{aligned} \tag{25}$$

and then

$$\begin{aligned}
\sigma_{m,i}^2(\tau) &= Var\left(U_{m,i}(\tau)\right) = E\left[\left(U_{m,i}(\tau)\right)^2\right] - \left(E\left[U_{m,i}(\tau)\right]\right)^2 = \\
&= \left(b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} {}_{Max(t,k)}p_{x_m}\right) - \left(b_m \sum_{t=1}^{\tau} (1+r)^{-t} {}_t p_{x_m}\right)^2 = \\
&= b_m^2 \left(\sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} {}_{Max(t,k)}p_{x_m} - \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-t} {}_t p_{x_m} (1+r)^{-k} {}_k p_{x_m}\right) = \\
&= b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} \left({}_{Max(t,k)}p_{x_m} - {}_t p_{x_m} {}_k p_{x_m}\right) = \\
&= b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau} (1+r)^{-(t+k)} \left(\prod_{s=0}^{Max\{t-1;k-1\}} p_{x_m+s} - \prod_{s=0}^{t-1+k-1} p_{x_m+s}\right).
\end{aligned} \tag{26}$$

Since the individuals in each group are mutually independent, then

$$\sigma_m^2(\tau) = Var(U_m(\tau)) = Var\left(\sum_{i=1}^{n_m} U_{m,i}(\tau)\right) = \sum_{i=1}^{n_m} \sigma_{m,i}^2(\tau) \tag{27}$$

and, as the groups (the 'sub portfolios') are themselves mutually independent,

$$\sigma^2(\tau) = Var(U(\tau)) = Var\left(\sum_{m=1}^{M} U_m(\tau)\right) = \sum_{m=1}^{M} \sigma_m^2(\tau). \tag{28}$$

To further compute the skewness coefficient $\gamma_{U_{m,i}(\tau)} = \frac{E\left[\left(U_{m,i}(\tau)-\mu_{m,i}(\tau)\right)^3\right]}{\sigma_{m,i}^3(\tau)}$ it is easier to use the relationship

$$E\left[\left(U_{m,i}(\tau) - \mu_{m,i}(\tau)\right)^3\right] = E\left[\left(U_{m,i}(\tau)\right)^3\right] - \mu_{m,i}^3(\tau) - 3\mu_{m,i}(\tau)\sigma_{m,i}^2(\tau),$$

as only the third raw moment of $U_{m,i}(\tau)$ is not yet known. Using the same computation approach as before, it follows that (see the full proof in Appendix 1)

$$E\left[\left(U_{m,i}(\tau)\right)^3\right] = E\left[\left(b_m \sum_{t=1}^{\tau} (1+r)^{-t} I(T_{m,i} > t)\right)^3\right] =$$

$$= b_m^3 E\left[\sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau} (1+r)^{-t} I(T_{m,i} > t)(1+r)^{-k} I(T_{m,i} > k)(1+r)^{-l} I(T_{m,i} > l)\right] =$$

$$= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau} (1+r)^{-(t+k+l)} E\left[I(T_{m,i} > t) I(T_{m,i} > k) I(T_{m,i} > l)\right] = \tag{29}$$

$$= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau} (1+r)^{-(t+k+l)} {}_{Max(t,k,l)}p_{x_m} =$$

$$= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau} (1+r)^{-(t+k+l)} \prod_{s=0}^{Max\{k-1;t-1;l-1\}} p_{x_m+s}.$$

Then,

$$E\left[\left(U_{m,i}(\tau) - \mu_{m,i}(\tau)\right)^3\right] =$$

$$= b_m^3 \left(\sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau} (1+r)^{-(t+k+l)} \left({}_{Max(t,k,l)}p_{x_m} - {}_t p_{x_m}\, {}_k p_{x_m}\, {}_l p_{x_m} - 3\, {}_l p_{x_m}\left({}_{Max(t,k)}p_{x_m} - {}_t p_{x_m}\, {}_k p_{x_m}\right)\right)\right).$$

and

$$\gamma_m(\tau) = \frac{E[(W_m(\tau) - \mu_m(\tau))^3]}{\sigma_m^3(\tau)} = \frac{\sum_{i=1}^{n_m} E\left[\left(W_{m,i}(\tau) - \mu_{m,i}(\tau)\right)^3\right]}{\sigma_m^3(\tau)} \tag{30}$$

$$\gamma(\tau) = \frac{E[(W(\tau)-\mu(\tau))^3]}{\sigma^3(\tau)} = \frac{\sum_{m=1}^{M} E[(W_m(\tau)-\mu_m(\tau))^3]}{\sigma^3(\tau)} \tag{31}$$

To close this chapter, a more precise description of the application in Chapter 4 follows.

(1) Calculate the ruin probability: the ruin probability studied in Frostig and Denuit (2009) is in line with equation (15) for the survival probability,

$$\widetilde{\Psi}(u,\tau) = P(\exists\, t \in \{0,1,\dots,\tau\}: U(t) < 0 | U(0) = u) = P(U(\tau) > u) =$$
$$= 1 - P(U(\tau) \leq u) = 1 - F_{U(\tau)}(u).$$

This represents the probability that the initial reserve u is not sufficient to accommodate the aggregate claim amount for the portfolio. The equality is proved taking into account the maximum aggregated loss criteria. Since payments are made under survival and survival to a specific time t implicates survival to all times $s \leq t$, then the probability that ruin occurs in any time during period $\{0,1,\dots,\tau\}$ ends up being exactly the same as it occurring at time τ. Thus, to determine the ruin probability, for the portfolio as a whole, there is 'only' the need to determine the distribution function of $U(\tau)$.

(2) Although the authors developed optimization problems to allocate the initial reserve u considering several constraints, as far as this work is concerned, the most significant one is the allocation of the initial reserve throughout the groups as to maximize the survival probability for the groups as a whole, problem P below.

$$Max\left\{\prod_{m=1}^{M} \widetilde{\Phi}(u_m,\tau)\right\}, \tag{P}$$

subject to:

$$\begin{cases} \sum_{m=1}^{M} u_m \leq u, & m = 1,2,\dots,M \\ \mu_m(\tau) \leq u_m, & m = 1,2,\dots,M \end{cases}.$$

The idea behind finding an optimal allocation of u is due with maximizing the survival probability at a most granular level as possible, using a variable the insurer may actually control, the reserve. This could be used to price the annuity, accordingly to a survival level. If an annuitant is expected to survive for longer (in this case a group of annuitants), then the price for the annuity must be augmented.

Chapter 4 – Case Study, the Application

4.1 The Portfolio

Consider a portfolio of 285 immediate whole life annuities. The annuitants are aged between $x_{min} = 60$ and $x_{Max} = 66$ years old, and since the annuities are already in payment, there are no premiums to be received and the only decrement to consider is death, with distinction between males and females. Figure 1 below presents the portfolio's membership, highlighting the divisions between the relevant factors for the following analysis: gender; age, in years; and benefit amount, in mu pa.

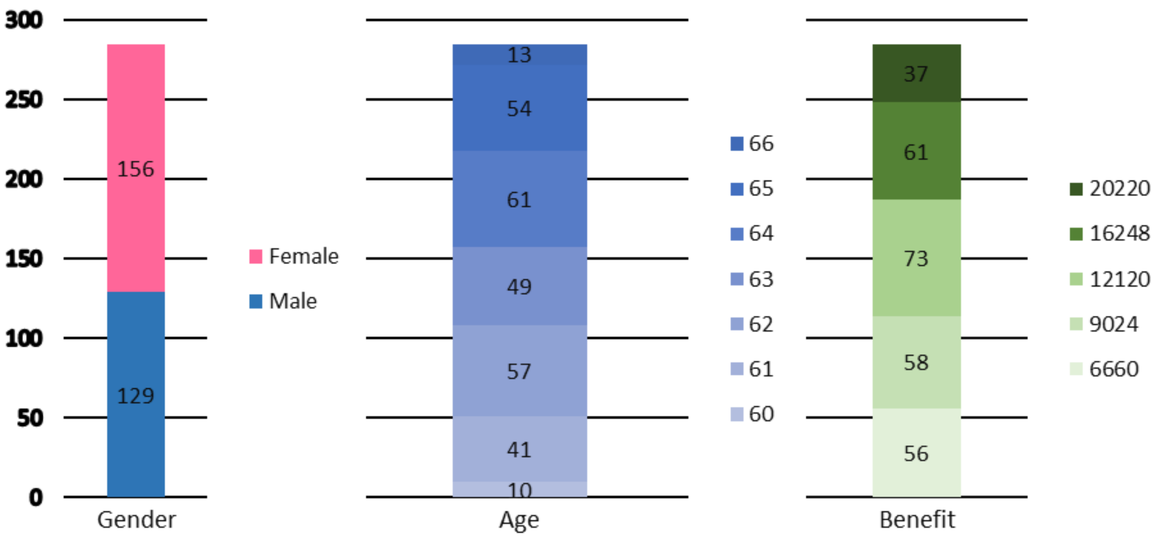


Figure 1 - Portfolio's Membership

This division leads to the categorisation of annuitants into $M = 7 \times 2 \times 5 = 70$ different groups, as described in section 3.2, such that annuitants in a specific group bear homogeneous characteristics. Thus, there are 70 homogeneous groups within this portfolio of heterogeneous lives. However, not all groups have annuitants. For instance, as may be observed by Table 2 (see full table in Appendix 2), in which the columns on the left represent male groups and the columns on the right female's, there are no annuitants, either men or women, aged 60 with a benefit amount of 9024 *pa*. The initial reserve for the whole portfolio is $u = 51{,}556{,}564$, which is the sum of each group's initial reserve.

Group	Age	Benefit	Number of Annuitants	Initial Reserve	Group	age	benefit	Number of Annuitants	Initial Reserve
M1	60	6660	1	103628	F1	60	6660	2	212201
M2	60	9024	0	0	F2	60	9024	0	0
⋮					⋮				
M35	66	20220	2	506196	F35	66	20220	0	0

Table 3 - Homogeneous Groups: some Data

Table 4 provides an overview of the main statistics observed in the portfolio.

		Males	Females	Total
Ages	Mean	63.17	63.11	63.14
	Standard Deviation	1.64	1.51	1.57
Benefit Amounts	Mean	11818.33	12793.69	12352.21
	Standard Deviation	4437.07	4464.82	4470.98

Table 4 - Main Portfolio Statistics

4.2 Technical Bases

The demographic assumptions include the use of two mortality tables: *S1PMA* for male annuitants and *S1PFA* for female annuitants. These tables are produced and provided by the Continuous Mortality Investigation (*CMI*), supported by the Institute and Faculty of Actuaries (*IFoA*)[1], using data from the *UK*, and are specifically designed to describe pensioners mortality experience. The limiting age for these tables is $\omega = 120$ and so $\tau = 120 - 60 = 60$ years.

The discount rate used to value the benefits may be established according to several scenarios, bearing different levels of prudence, but upon trials only the most relevant are considered here: a term structure of *UK* nominal interest rates (*TSIR*), published by the Bank of England (*BoE*) in 15/06/2015[2], which yields a single equivalent rate (*SER*) of about 2.14% (Scenario 1); and a fixed rate (*FR*) equal to 3.5% (Scenario 2). Scenario 1 applies the *TSIR*

[1] http://www.actuaries.org.uk/research-and-resources/pages/continuous-mortality-investigation
[2] http://www.bankofengland.co.uk /statistics/Pages/yieldcurve/default.aspx

provided by the *BoE*, year by year, until the maximum projected interest rate, for a maturity of 25 years, is reached. Thereafter, the *SER*, computed as $SER = \left(\prod_{t=1}^{25} TSIR_t^{\frac{1}{25}} - 1\right) * 100 \approx 2.14\%$, is applied. The difference between these two assumptions is determined by the fact that the *TSIR* produces a more realistic scenario, whereas the *FR* requires the need for specific investment strategies or growth guarantees.

4.3 Results

4.3.1 The distributions of reserves

With the categorization and assumptions above, it was possible to determine the distribution of $U(\tau)$, in line with the models described, and finally compute the ruin probabilities associated with this portfolio.

For simplicity, the first approach consisted of studying the empirical distribution of the individual claim amount present value. The empirical pdf of $U_{m,i}(\tau)$ is determined for each specific individual in group m, considering any age or gender and ignoring, at first, the benefit amount. This means that the benefit amount considered is 1 *mu pa* and it is possible to obtain the *pdf* of $U_{m,i}(\tau)$ by using this assumption since this *rv* does not depend on the amount of each benefit - it is constant year after year.

The formula for the *pdf* of $U_{m,i}(\tau)$ is $f_{U_{m,i}(\tau)}(x) = P\left(U_{m,i}(\tau) = x\right) = {}_tp_{x_m} q_{x_m+t}$, for $x = b_m \sum_{t=1}^{\tau}(1+r_t)^{-t}$ and $t = 1, 2, \ \dots, \tau$. Table 7 shows the computations for a female aged 65 years-old, under both economic scenarios. Such an annuitant was chosen for representing one of the most populated groups within the portfolio. The *pdf* depends only on the year and the mortality assumptions, though the payments in each year depend on the economic assumptions, as shown in this table.

	Scenario 1			Scenario 2					
t	**Interest Rate** r_t	**Yearly Unit Payment** $(1+r_t)^{-t}$	**Accumulated Payment** $\sum_{s=0}^{t}(1+r_s)^{-s}$	**Interest Rate** r_t	**Yearly Unit Payment** $(1+r_t)^{-t}$	**Accumulated Payment** $\sum_{s=0}^{t}(1+r_s)^{-s}$	${}_tp_{x_m}$	p_{x_m+t}	***pdf*** ${}_tp_{x_m}q_{x_m+t}$
0	0,00%	0,00	0,00	3,50%	0,00	0,00	1,0000	1,0000	0,0079
1	0,41%	1,00	1,00	3,50%	0,97	0,97	0,9921	0,9921	0,0086
2	0,60%	0,99	1,98	3,50%	0,93	1,90	0,9834	0,9913	0,0094
3	0,90%	0,97	2,96	3,50%	0,90	2,80	0,9740	0,9904	0,0104
4	1,17%	0,95	3,91	3,50%	0,87	3,67	0,9636	0,9894	0,0115
5	1,41%	0,93	4,84	3,50%	0,84	4,52	0,9522	0,9881	0,0127
6	1,61%	0,91	5,75	3,50%	0,81	5,33	0,9395	0,9867	0,0141
7	1,78%	0,88	6,64	3,50%	0,79	6,11	0,9254	0,9850	0,0156
8	1,92%	0,86	7,50	3,50%	0,76	6,87	0,9098	0,9831	0,0174
9	2,04%	0,83	8,33	3,50%	0,73	7,61	0,8924	0,9809	0,0192
10	2,15%	0,81	9,14	3,50%	0,71	8,32	0,8732	0,9784	0,0213
11	2,25%	0,78	9,92	3,50%	0,68	9,00	0,8519	0,9756	0,0235

12	2,33%	0,76	10,68	3,50%	0,66	9,66	0,8285	0,9725	0,0258
13	2,41%	0,73	11,41	3,50%	0,64	10,30	0,8027	0,9689	0,0282
14	2,48%	0,71	12,12	3,50%	0,62	10,92	0,7745	0,9649	0,0306
15	2,54%	0,69	12,81	3,50%	0,60	11,52	0,7439	0,9605	0,0331
16	2,60%	0,66	13,47	3,50%	0,58	12,09	0,7109	0,9556	0,0355
17	2,65%	0,64	14,11	3,50%	0,56	12,65	0,6754	0,9501	0,0378
18	2,69%	0,62	14,73	3,50%	0,54	13,19	0,6376	0,9440	0,0399
19	2,73%	0,60	15,33	3,50%	0,52	13,71	0,5976	0,9373	0,0419
20	2,76%	0,58	15,91	3,50%	0,50	14,21	0,5558	0,9300	0,0434
21	2,79%	0,56	16,48	3,50%	0,49	14,70	0,5124	0,9219	0,0446
22	2,81%	0,54	17,02	3,50%	0,47	15,17	0,4677	0,9129	0,0453
23	2,82%	0,53	17,55	3,50%	0,45	15,62	0,4224	0,9031	0,0455
24	2,83%	0,51	18,06	3,50%	0,44	16,06	0,3769	0,8923	0,0451
25	2,84%	0,50	18,56	3,50%	0,42	16,48	0,3319	0,8805	0,0440
26	2,14%	0,58	19,13	3,50%	0,41	16,89	0,2879	0,8674	0,0423
27	2,14%	0,56	19,70	3,50%	0,40	17,29	0,2456	0,8531	0,0399
28	2,14%	0,55	20,25	3,50%	0,38	17,67	0,2057	0,8374	0,0370
29	2,14%	0,54	20,79	3,50%	0,37	18,04	0,1687	0,8201	0,0336
30	2,14%	0,53	21,32	3,50%	0,36	18,39	0,1351	0,8011	0,0297
31	2,14%	0,52	21,84	3,50%	0,34	18,74	0,1054	0,7804	0,0254
32	2,14%	0,51	22,35	3,50%	0,33	19,07	0,0800	0,7590	0,0210
33	2,14%	0,50	22,85	3,50%	0,32	19,39	0,0590	0,7371	0,0168
34	2,14%	0,49	23,33	3,50%	0,31	19,70	0,0422	0,7151	0,0129
35	2,14%	0,48	23,81	3,50%	0,30	20,00	0,0292	0,6930	0,0096
36	2,14%	0,47	24,28	3,50%	0,29	20,29	0,0196	0,6711	0,0069
37	2,14%	0,46	24,74	3,50%	0,28	20,57	0,0127	0,6493	0,0047
38	2,14%	0,45	25,18	3,50%	0,27	20,84	0,0080	0,6279	0,0031
39	2,14%	0,44	25,62	3,50%	0,26	21,10	0,0049	0,6068	0,0020
40	2,14%	0,43	26,05	3,50%	0,25	21,36	0,0028	0,5864	0,0012
41	2,14%	0,42	26,47	3,50%	0,24	21,60	0,0016	0,5665	0,0007
42	2,14%	0,41	26,88	3,50%	0,24	21,83	0,0009	0,5472	0,0004
43	2,14%	0,40	27,29	3,50%	0,23	22,06	0,0005	0,5287	0,0002
44	2,14%	0,39	27,68	3,50%	0,22	22,28	0,0002	0,5110	0,0001
45	2,14%	0,39	28,07	3,50%	0,21	22,50	0,0001	0,4940	0,0001
46	2,14%	0,38	28,44	3,50%	0,21	22,70	0,0001	0,4778	0,0000
47	2,14%	0,37	28,81	3,50%	0,20	22,90	0,0001	1,0000	0,0000
48	2,14%	0,36	29,18	3,50%	0,19	23,09	0,0001	1,0000	0,0000
49	2,14%	0,35	29,53	3,50%	0,19	23,28	0,0001	1,0000	0,0000
50	2,14%	0,35	29,88	3,50%	0,18	23,46	0,0001	1,0000	0,0000
51	2,14%	0,34	30,22	3,50%	0,17	23,63	0,0001	1,0000	0,0000
52	2,14%	0,33	30,55	3,50%	0,17	23,80	0,0001	1,0000	0,0000
53	2,14%	0,33	30,88	3,50%	0,16	23,96	0,0001	1,0000	0,0000
54	2,14%	0,32	31,20	3,50%	0,16	24,11	0,0001	1,0000	0,0000
55	2,14%	0,31	31,51	3,50%	0,15	24,26	0,0001	1,0000	0,0001
56	2,14%	0,31	31,82	3,50%	0,15	24,41	0,0000	0,0000	0,0000
57	2,14%	0,30	32,11	3,50%	0,14	24,55	0,0000	0,0000	0,0000
58	2,14%	0,29	32,41	3,50%	0,14	24,69	0,0000	0,0000	0,0000
59	2,14%	0,29	32,70	3,50%	0,13	24,82	0,0000	0,0000	0,0000
60	2,14%	0,28	32,98	3,50%	0,13	24,94	0,0000	0,0000	0,0000

Table 5 - Empirical pdf for a Female aged 65 (Scenarios 1 and 2)

The *pdf* for each combination of gender and age in the portfolio is presented in Appendix 3. Figures 3 and 4 present the graphical representation of the *pdf* of $U_{m,i}(\tau)$, considering the same exemplifying annuitant, under the two economic scenarios.

Figure 2 - Empirical pdf for a Female aged 65 (Scenario 1)

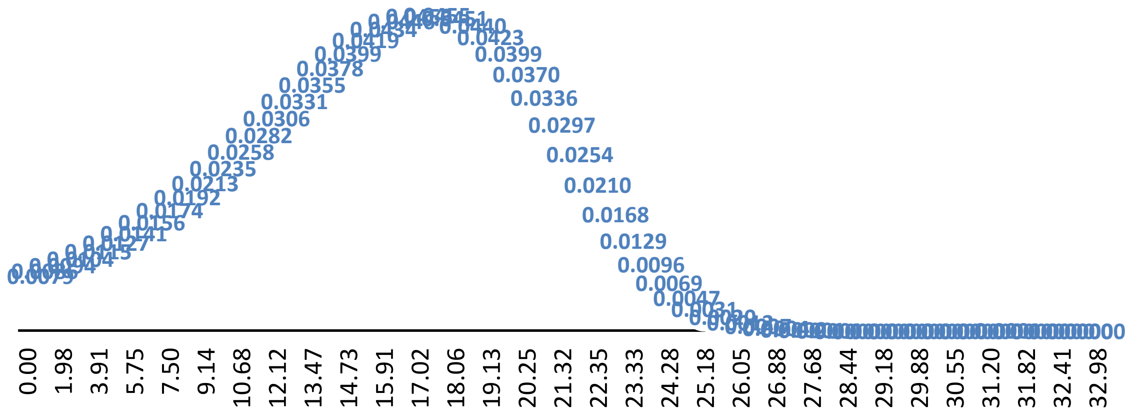
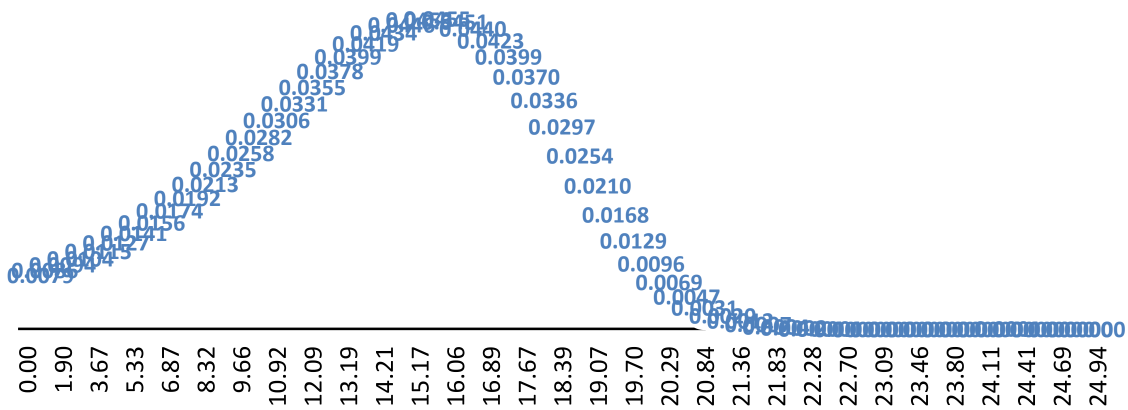


Figure 3 - Empirical pdf for a Female aged 65 (Scenario 2)



In these figures, there is a slight resemblance with the bell shape of the Normal pdf, irrespectively of the economic assumption chosen, although they do not show a high degree of symmetry, especially as far as the tails are concerned. In fact, there is a consistent skewness across ages observed (right for females and left for males). Though an assumption for the individual distribution is unnecessary (the actual *pdf* has been computed), since groups are the sum of individuals and the portfolio is the sum of groups and, furthermore, the sum of Normal distributed *rvs* is itself a Normal distributed *rv*, then the Normal distribution may work as a fit for the portfolio as a whole. The similarity with the Normal pdf bell shape is to a certain extent confirmed when drawing the *pdf* of a Normal *rv* with the same mean and standard deviation as $U_{m,i}(\tau)$, as seen in Figures 5 and 6. Table 6 presents the relevant parameters for this analysis, which were computed using the *pdf* derived above:

$$\mu_{m,i}(\tau) = \sum_{t=0}^{\tau} \left(\sum_{s=0}^{t} (1 + r_s)^{-s} \right) {}_t p_{x_m} q_{x_m + t}, \tag{32}$$

$$\sigma_{m,i}^2(\tau) = \sum_{t=0}^{\tau}\left(\sum_{s=0}^{t}(1+r_s)^{-s}\right)^2 {}_t p_{x_m} q_{x_m+t} - \mu_{m,i}^2(\tau) \tag{33}$$

and

$$\gamma_{m,i}^3 = \frac{E\left[\left(U_{m,i}(\tau)-\mu_{m,i}(\tau)\right)^3\right]}{\sigma_{m,i}^3(\tau)} = \frac{\sum_{t=0}^{\tau}(\sum_{s=0}^{t}(1+r_s)^{-s})^3 {}_t p_{x_m} q_{x_m+t} - \mu_{m,i}^3(\tau) - 3\mu_{m,i}(\tau)\sigma_{m,i}^2(\tau)}{\sigma_{m,i}^3(\tau)}. \tag{34}$$

	$\mu_{m,i}(\tau)$	$\sigma_{m,i}(\tau)$	$\gamma_{m,i}(\tau)$
Scenario 1	15,43	5,43	-0,6999
Scenario 2	13,63	4,56	-0,8800

Table 6 - Parameters for a Female aged 65

Figure 4 - Normal pdf for a Female aged 65 (Scenario 1)

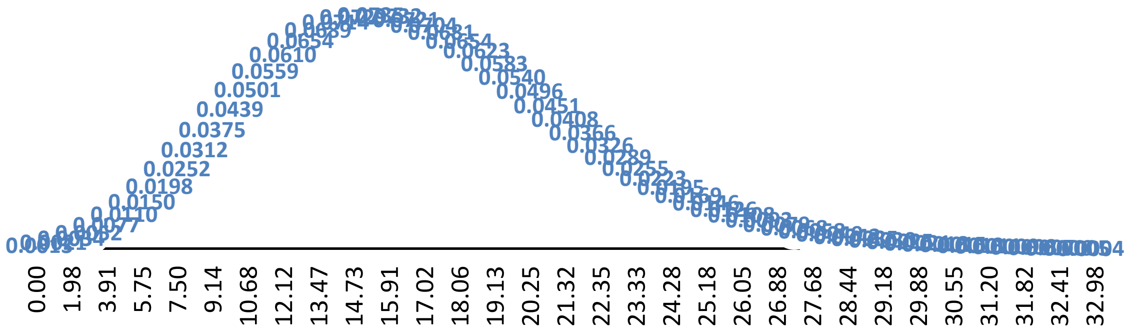
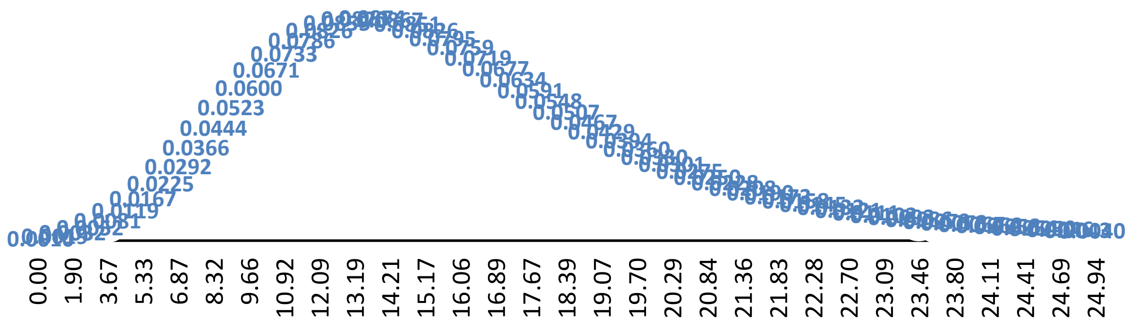


Figure 5 - Normal pdf for a Female aged 65 (Scenario 2)



A more formal approach to assess the goodness of fit regarding the Normal distribution is to perform a $\mathcal{X}^2$ goodness of fit test. The $\mathcal{X}^2$ test provided a p-value very close to 1 for all annuitants, considering the relevant groups in terms of gender and age only. Thus, there is no evidence against the Normal distribution. As a result, one may consider the Normal approximation as not being a bad approximation to the *pdf* of $U_{m,i}(\tau)$ for a generalization to be carried out on its grounds, though, as age increases, the tails of the distribution are more and more difficult to fit, since probabilities of survival are smaller and smaller. The parameters for each combination of gender and age in the portfolio are presented in Appendix 3, as well.

As mentioned in 3.2, the exact distribution of $U_m(\tau)$ and $U(\tau)$ using the De Pril recursion method was not obtained. The alternative would be to apply the convolutions method, which is an even more time consuming approach, only possible using a software other than Excel. So the moment-based approximations are preferred. The mean, standard deviation and skewness coefficient of $U_m(\tau)$ and $U(\tau)$ are obtained considering the moments computed for each individual using the empirical *pdf* of $U_{m,i}(\tau)$, as presented in Tables 7 and 8 and under equations (32), (33) and (34), using, respectively, (23), (27) and (30), (24), (28) and (31). The parameters for $U(\tau)$ are presented in Table 7 below, once again considering the two discount rate scenarios.

	Scenario 1			Scenario 2		
	$\mu(\tau)$	$\sigma(\tau)$	$\gamma(\tau)$	$\mu(\tau)$	$\sigma(\tau)$	$\gamma(\tau)$
$\tau = 60$	55,166,999	1,214,373	-0,0321	48,599,061	1,015,521	-0,0399

Table 7 - The Parameters for the whole Portfolio

As already mentioned, there is a certain level of skewness, which is consistent across the portfolio. And though it is not significant to rule out the Normal approximation, it is in the case of the *NP* and the translated Gamma approximations. For all the relevant ages and genders considered, meaning for each group, the skewness coefficient is negative, which precludes the use of these approximations. The *NP*, standard or adjusted, as well as the translated Gamma, once applied, could provide an insight into why the Normal approximation is not a good fit, especially if the tails are concerned.

However, when performing computations to determine the adjusted *NP* approximation, a disappointing discovery presented. The parameters are quite similar to those of the standard *NP* (Table 8), yielding non-treatable values and so it is not possible to use any of these approximations as a mean to compare with the Normal results.

	a	b
Standard NP	1	$\frac{\gamma(\tau)}{6} \approx$-0.01115

Adjusted NP	≈ 0.99987	≈ -0.01134
Difference	$\approx -0.01\%$	1.66%

Table 8 - NP Approximation Parameters

Nevertheless, for other age ranges it could be used. As shown in Table 9, for a fixed interest rate assumption, as age increases, the probabilities of survival decrease and so the skewness coefficient for $U_{m,i}(\tau)$ increases. For men, the effect is recorded earlier since men are historically less likely to survive a given period of time than women.

		45	50	55	60	65	70	75	80	85	90	95
Skewness Coefficient for $\tau = 60$ and $r = 0\%$	Males	-0,79	-0,63	-0,46	-0,29	-0,11	0,10	0,32	0,56	0,82	1,11	1,48
	Females	-0,99	-0,81	-0,63	-0,44	-0,25	-0,06	0,15	0,38	0,64	0,94	1,37

Table 9 - Skewness Coefficient

On the other hand, for a fixed age, as the interest rate decreases, the skewness coefficient increases. So for portfolios in these conditions, older ages or lower interest rates, the application of the *NP* or the translated Gamma approximations would be applicable.

4.3.2 The ruin probabilities

Regarding the ruin probability, Table 10 includes the results obtained for both economic scenarios, using the Normal approximation.

	Scenario 1		Scenario 2	
	$\tilde{\Phi}(u,\tau)$	$\tilde{\Psi}(u,\tau)$	$\tilde{\Phi}(u,\tau)$	$\tilde{\Psi}(u,\tau)$
$\tau = 60$	0,1474%	99,8526%	99,8206%	0,1794%

Table 10 - Ruin Probabilities using the Normal Approximation

The reason for this enormous difference is the fact that yearly interest rates are essentially much lower under the Nominal *UK TSIR* (thus why the corresponding *SER* is low) when compared to a fixed annual assumption of 3.5%. This shows the high dependence of ruin probabilities on the discount factor chosen. The data was provided with no information regarding the interest rate assumption taken when computing the initial reserve. However, it is possible to conclude that, considering a usual limiting age, the interest rate must be high, by the standards of current economic situations. Scenario 2 has proven to be efficient when determining an acceptable level of ruin probability. On the other hand, under Scenario 1, by evaluating the ruin probabilities for different time frames of payments, it is possible to conclude that if the insurer were to assume the Nominal *UK TSIR* to compute the reserve it would suffice only to about a 30 to 40-year period, since $\widetilde{\Psi}(u, 31) = 0.0001\%$, $\widetilde{\Psi}(u, 35) = 41.99\%$ and $\widetilde{\Psi}(u, 39) = 99.27\%$

4.3.3 Optimal allocation of the initial reserve amongst the groups

The Normal approximation performs quite well. So the last step to take is to apply the optimization problem (P) to allocate as best as possible the initial reserve through the various groups.

The results presented in Table 11 refer to the scenario which bears a lower ruin probability within the time frame of payments considered, the $\tau = 60$ years, which is Scenario 2, using the fixed rate assumption of 3.5% pa.

Group	Age	Benefit	Initial Reserve	Optimized Reserve	Difference	Group	age	benefit	Initial Reserve	Optimized Reserve	Difference
M1	60	6660	103628	101988,25	-1,58%	F1	60	6660	212201	210561,25	-0,77%
M2	60	9024	0	0,00	-	F2	60	9024	0	0,00	-
M3	60	12120	728073	726433,25	-0,23%	F3	60	12120	404541	402901,25	-0,41%
M4	60	16248	233503	239079,80	2,39%	F4	60	16248	0	0,00	-
M5	60	20220	0	0,00	-	F5	60	20220	0	0,00	-
M6	61	6660	400238	398598,25	-0,41%	F6	61	6660	330647	329007,25	-0,50%
M7	61	9024	403346	401706,25	-0,41%	F7	61	9024	1024827	1023187,25	-0,16%
M8	61	12120	891035	889395,25	-0,18%	F8	61	12120	1385780	1384140,25	-0,12%
M9	61	16248	1200079	1198439,25	-0,14%	F9	61	16248	781867	780227,25	-0,21%
M10	61	20220	285627	283987,25	-0,57%	F10	61	20220	984022	982382,25	-0,17%
M11	62	6660	687179	685539,25	-0,24%	F11	62	6660	518068	516428,25	-0,32%
M12	62	9024	763791	762151,25	-0,21%	F12	62	9024	572228	570588,25	-0,29%
M13	62	12120	719797	718157,25	-0,23%	F13	62	12120	1891741	1890101,25	-0,09%
M14	62	16248	932272	930632,25	-0,18%	F14	62	16248	1786833	1785193,25	-0,09%
M15	62	20220	1176357	1174717,25	-0,14%	F15	62	20220	1914801	1913161,25	-0,09%
M16	63	6660	551876	550236,25	-0,30%	F16	63	6660	207863	206223,25	-0,79%
M17	63	9024	496392	494752,25	-0,33%	F17	63	9024	1387714	1386074,25	-0,12%
M18	63	12120	856149	854509,25	-0,19%	F18	63	12120	1122848	1121208,25	-0,15%
M19	63	16248	448197	446557,25	-0,37%	F19	63	16248	1717229	1715589,25	-0,10%
M20	63	20220	569898	568258,25	-0,29%	F20	63	20220	1548626	1546986,25	-0,11%
M21	64	6660	363838	362198,25	-0,45%	F21	64	6660	494030	492390,25	-0,33%
M22	64	9024	733824	732184,25	-0,22%	F22	64	9024	1095922	1094282,25	-0,15%
M23	64	12120	1660377	1658737,25	-0,10%	F23	64	12120	1593510	1591870,25	-0,10%
M24	64	16248	1501096	1499456,25	-0,11%	F24	64	16248	1694394	1692754,25	-0,10%
M25	64	20220	535030	533390,25	-0,31%	F25	64	20220	868974	867334,25	-0,19%
M26	65	6660	605717	604077,25	-0,27%	F26	65	6660	576697	575057,25	-0,28%
M27	65	9024	356560	354920,25	-0,46%	F27	65	9024	647125	645485,25	-0,25%
M28	65	12120	951827	950187,25	-0,17%	F28	65	12120	553189	551549,25	-0,30%
M29	65	16248	1111665	1110025,25	-0,15%	F29	65	16248	2388620	2386980,25	-0,07%
M30	65	20220	782728	781088,25	-0,21%	F30	65	20220	1744868	1743228,25	-0,09%
M31	66	6660	336628	334988,25	-0,49%	F31	66	6660	0	0,00	-
M32	66	9024	113415	111775,25	-1,45%	F32	66	9024	121148	119508,25	-1,35%
M33	66	12120	144331	149907,80	3,86%	F33	66	12120	161827	160187,25	-1,01%
M34	66	16248	0	0,00	-	F34	66	16248	673755	672115,25	-0,24%
M35	66	20220	506196	504556,25	-0,32%	F35	66	20220	0	0,00	-

Table 11 - Optimization of Initial Reserves

As one may observe from Table 11, the initial reserve is roughly well spread over the different groups in terms of survival maximization. Mostly the changes between the optimized and the initial reserve are small, with the ones above 1% difference, in absolute value, flagged in red. Overall, the actual reserve allocated after the optimization, $u^{Opt} = 51{,}469{,}332$, is smaller than the initial by about -0.17%.

4.4 Comparison with Denuit and Frostig (2009)

Frostig and Denuit (2009) worked with a portfolio where all annuitants were aged 65 years old, there were four different mortality models and a fixed interest rate equal to 3% per year, computing the ruin probability within 45 years, i. e., assuming ruin probability until the limiting age, which for the life tables the authors used is 110 years.

Comparatively, the present thesis shows some developments: several ages and two economic scenarios are now explored and the setup is easily extensive to other mortality assumptions. Although it really does not come as a surprise, it is always important to realize the high dependence of results on the discount rate used, which is in fact a very central aspect to consider, since the provider of the annuities must know which rate guarantees at least a break-even point.

Finally, this work carried a more granular study on the possible methods to be used and how these may be applied, setting a simple formula to compute the Normal parameters and the skewness coefficient, which the authors had not developed.

Chapter 5 - Conclusions

This thesis followed closely the procedures set in Frostig and Denuit (2009) in an attempt to study how the individual risk model could be applied to the problem of minimizing ruin probabilities for portfolios of life annuity contracts. After the necessary background on both topics is developed, the several possible approaches were analysed. Namely, the primary requirement for the development of the work, fitting a distribution to the reserve *rv* was thoroughly studied as to assess the best option to apply, amongst the ones which could actually be applied.

With such goal set, through the course of this work, two major step backs presented themselves and led to the pursuit of alternative methods. The first step back refers to the initial probability for the pdf of $U(\tau)$ as defined in the De Pril recursion formula. The fact that this initial condition is a number indistinguishable from zero precludes the use of this method and any other method with such an initial condition, as is the case of the Dhaene and Vandebrok or the Kornya and Hipp. The most discouraging fact on this matter is that this is a problem affecting the whole range of ages. The reason for the null initial condition is the fact that the De Pril method was setup for a contrary situation, i. e., it is most valid when there are no payments to be made if every policyholder survives, since payments are usually assumed to be paid under death. When considering life annuities, the case is the reverse. Though an approach to surpass this problem was discovered timeliness of work did not permit its application.

The second step back refers to the *NP* approximation. The fact that the skewness coefficient is negative for all relevant ages, from both an annuitant and a portfolio perspective, precludes the use of this approximation. The alternative is to use the adjusted *NP* approximation, which is defined for a wider range of the skewness coefficient' values. However, another discouraging fact, the parameters produced by the adjusted *NP* approach are awfully similar to the ones produced by the standard *NP* approach. As a result, the *NP* approximation cannot be applied at all. Since the results from the Translated Gamma approximation are close to those of the *NP*, then the Gamma is excluded as well.

Basically, the only method left is the one which has shown to produce results more accurately than at first was expected. The Normal approximation, though not perfectly, fits well the distribution of the aggregate claim payments and permits the computation of ruin probabilities. In fact, this method was the only one used in Frostig and Denuit (2009). The other methods were merely referred. Other methods, apart from the Normal could be developed. A good approach would be to apply the alternative approach and the time-consuming convolution method and determine what distributions or simpler methods in line with the De Pril setup, could apply, considering the specific features of life annuity contracts. This process would certainly involve the use of different software to perform calculations for their complexity.

Further works could follow closely the household perspective, eventually using some of the features applicable to the individual risk model. The requirement on this matter is to try to determine a way to compute the probability of lifetime ruin for an individual to be used as an advisory tool for possible costumers of an insurance company, members of a pension scheme or individuals in general, regarding the features expected of those plans and how they adapt to the individual's characteristics. This is a daring time-consuming proposition, only possible after years of research and continued work on the subject. A good approach would be to study the literature review presented for the household perspective. Nevertheless, this is certainly a topic worth developing in such a way that even more meaningful conclusions may be obtained.

The models above may be developed and applied to several daily situations helping in decision making and risk management for companies, as well as households. For these reasons, the continued study and development of the concepts and models described is imperative.

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Appendix 1

The formulae presented in section 3.2.1 for the skewness coefficient was originally developed to apply the NP approximation. Though this was not possible in the end, the formulae still holds and may be found below.

The standard formula for the skewness coefficient is $\gamma_{U_{m,i}(\tau)} = \frac{E\left[\left(U_{m,i}(\tau)-\mu_{m,i}(\tau)\right)^3\right]}{\sigma_{m,i}^3(\tau)}$. The following relationship for the 3rd central moment is easier to apply since the mean and variance have already been computed:

$$E\left[\left(U_{m,i}(\tau) - \mu_{m,i}(\tau)\right)^3\right] = E\left[\left(U_{m,i}(\tau)\right)^3\right] - \mu_{m,i}^3(\tau) - 3\mu_{m,i}(\tau)\sigma_{m,i}^2(\tau).$$

The outstanding parameter is the 3rd raw moment computed as:

$$\begin{aligned} E\left[\left(U_{m,i}(\tau)\right)^3\right] &= E\left[\left(b_m \sum_{t=1}^{\tau}(1+r)^{-t} I\left(T_{m,i} > t\right)\right)^3\right] = \\ &= b_m^3 E\left[\sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-t} I\left(T_{m,i} > t\right)(1+r)^{-k} I\left(T_{m,i} > k\right)(1+r)^{-l} I\left(T_{m,i} > l\right)\right] = \\ &= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-(t+k+l)} E\left[I\left(T_{m,i} > t\right) I\left(T_{m,i} > k\right) I\left(T_{m,i} > l\right)\right] = \\ &= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-(t+k+l)}\,{}_{Max(t,k,l)}p_{x_m} = \\ &= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-(t+k+l)} \prod_{s=0}^{Max\{k-1;t-1;l-1\}} p_{x_m+s}. \end{aligned} \tag{29}$$

or as an alternative, perhaps more elegant way (Centeno), following the fact that

$$\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} a_i{}^3 + 3\sum_{i=1}^{n} a_i{}^2 \sum_{j>i} a_j + 3\sum_{i=1}^{n} a_i \sum_{j>i} a_j{}^2 + 6\sum_{i=1}^{n}\sum_{j>i}\sum_{k>j} a_i a_j a_k$$

then

$$E\left[\left(U_{m,i}(\tau)\right)^3\right] = E\left[\left(b_m \sum_{t=1}^{\tau}(1+r)^{-t} I\left(T_{m,i} > t\right)\right)^3\right] =$$

$$= b_m^3 E\left[\sum_{t=1}^{\tau} I(T_{m,i} > t)(1+r)^{-3t}\right.$$

$$+ 3\sum_{t=1}^{\tau}\sum_{l>t} I(T_{m,i} > l)(1+r)^{-2t-l}$$

$$\left.+ 3\sum_{t=1}^{\tau}\sum_{l>t} I(T_{m,i} > l)(1+r)^{-t-2l} + 6\sum_{t=1}^{\tau}\sum_{l>t}\sum_{k>l} I(T_{m,i} > k)(1+r)^{-t-l-k}\right] =$$

$$= b_m^3 \cdot E\left[\sum_{t=1}^{\tau}(1+r)^{-3t}\, {}_t p_{x_m} + 3\sum_{t=1}^{\tau}\sum_{l>t}(1+r)^{-2t-l}\, {}_l p_{x_m} + 3\sum_{t=1}^{\tau}\sum_{l>t}(1+r)^{-t-2l}\, {}_l p_{x_m} + 6\sum_{t=1}^{\tau}\sum_{l>t}\sum_{k>l}(1+r)^{-t-l-k}\, {}_k p_{x_m}\right].$$

Then,

$$E\left[\left(U_{m,i}(\tau) - \mu_{m,i}(\tau)\right)^3\right] =$$

$$= b_m^3 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-(t+k+l)} \prod_{s=0}^{Max\{k-1;t-1;l-1\}} p_{x_m+s} - \left(b_m \sum_{t=1}^{\tau}(1+r)^{-t}\prod_{s=0}^{t-1} p_{x_m+s}\right)^3$$

$$- 3\left(b_m \sum_{t=1}^{\tau}(1+r)^{-t}\prod_{s=0}^{t-1} p_{x_m+s}\right)\left(b_m^2 \sum_{t=1}^{\tau}\sum_{k=1}^{\tau}(1+r)^{-(t+k)}\left(\prod_{s=0}^{Max\{t-1;k-1\}} p_{x_m+s} - \prod_{s=0}^{t-1+k-1} p_{x_m+s}\right)\right)$$

$$=$$

$$= b_m^3\left(\sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-(t+k+l)}\left({}_{Max(t,k,l)}p_{x_m} - {}_t p_{x_m}\, {}_k p_{x_m}\, {}_l p_{x_m} - 3\, {}_l p_{x_m}\left({}_{Max(t,k)}p_{x_m} - {}_t p_{x_m}\, {}_k p_{x_m}\right)\right)\right).$$

Finally, the skewness coefficient for group m is obtained using the fact that individuals in each group are mutually independent:

$$\gamma_m(\tau) = \frac{E\left[\left(W_m(\tau) - \mu_m(\tau)\right)^3\right]}{\sigma_m^3(\tau)} = \frac{\sum_{i=1}^{n_m} E\left[\left(W_{m,i}(\tau) - \mu_{m,i}(\tau)\right)^3\right]}{\sigma_m^3(\tau)} =$$

$$= \left(b_m^3\left(\sum_{t=1}^{\tau}\sum_{k=1}^{\tau}\sum_{l=1}^{\tau}(1+r)^{-(t+k+l)}\left({}_{Max(t,k,l)}p_{x_m} - {}_t p_{x_m}\, {}_k p_{x_m}\, {}_l p_{x_m} - 3\, {}_l p_{x_m}\left({}_{Max(t,k)}p_{x_m} - {}_t p_{x_m}\, {}_k p_{x_m}\right)\right)\right)\right) \Big/ \left(\sum_{i=1}^{n_m}\sigma_{m,i}^2(\tau)\right)^{\frac{3}{2}} \quad (30)$$

$$\gamma(\tau) = \frac{E[(W(\tau) - \mu(\tau))^3]}{\sigma^3(\tau)} = \frac{\sum_{m=1}^{M} E[(W_m(\tau) - \mu_m(\tau))^3]}{\sigma^3(\tau)} \quad (31)$$

Appendix 2

Group	Age	Benefit	Number of Annuitants	Initial Reserve	Group	Age	Benefit	Number of Annuitants	Initial Reserve
M1	60	6660	1	103628	F1	60	6660	2	212201
M2	60	9024	0	0	F2	60	9024	0	0
M3	60	12120	4	728073	F3	60	12120	2	404541
M4	60	16248	1	233503	F4	60	16248	0	0
M5	60	20220	0	0	F5	60	20220	0	0
M6	61	6660	4	400238	F6	61	6660	3	330647
M7	61	9024	3	403346	F7	61	9024	7	1024827
M8	61	12120	5	891035	F8	61	12120	7	1385780
M9	61	16248	5	1200079	F9	61	16248	3	781867
M10	61	20220	1	285627	F10	61	20220	3	984022
M11	62	6660	7	687179	F11	62	6660	5	518068
M12	62	9024	6	763791	F12	62	9024	4	572228
M13	62	12120	4	719797	F13	62	12120	10	1891741
M14	62	16248	4	932272	F14	62	16248	7	1786833
M15	62	20220	4	1176357	F15	62	20220	6	1914801
M16	63	6660	6	551876	F16	63	6660	2	207863
M17	63	9024	4	496392	F17	63	9024	10	1387714
M18	63	12120	5	856149	F18	63	12120	6	1122848
M19	63	16248	2	448197	F19	63	16248	7	1717229
M20	63	20220	2	569898	F20	63	20220	5	1548626
M21	64	6660	4	363838	F21	64	6660	5	494030
M22	64	9024	6	733824	F22	64	9024	8	1095922
M23	64	12120	10	1660377	F23	64	12120	9	1593510
M24	64	16248	7	1501096	F24	64	16248	7	1694394
M25	64	20220	2	535030	F25	64	20220	3	868974
M26	65	6660	7	605717	F26	65	6660	6	576697
M27	65	9024	3	356560	F27	65	9024	5	647125
M28	65	12120	6	951827	F28	65	12120	3	553189
M29	65	16248	5	1111665	F29	65	16248	10	2388620
M30	65	20220	3	782728	F30	65	20220	6	1744868
M31	66	6660	4	336628	F31	66	6660	0	0
M32	66	9024	1	113415	F32	66	9024	1	121148
M33	66	12120	1	144331	F33	66	12120	1	161827
M34	66	16248	0	0	F34	66	16248	3	673755
M35	66	20220	2	506196	F35	66	20220	0	0

Table 12 - Homogeneous Groups with Numbers

Appendix 3

	Males							Females						
t	**60**	**61**	**62**	**63**	**64**	**65**	**66**	**60**	**61**	**62**	**63**	**64**	**65**	**66**
0	0,0069	0,0075	0,0083	0,0091	0,0101	0,0112	0,0125	0,0057	0,0060	0,0064	0,0068	0,0073	0,0079	0,0087
1	0,0075	0,0082	0,0090	0,0100	0,0111	0,0124	0,0138	0,0060	0,0063	0,0067	0,0073	0,0079	0,0086	0,0095
2	0,0081	0,0090	0,0099	0,0110	0,0123	0,0137	0,0153	0,0063	0,0067	0,0072	0,0078	0,0086	0,0094	0,0105
3	0,0089	0,0099	0,0109	0,0122	0,0135	0,0151	0,0168	0,0067	0,0072	0,0078	0,0085	0,0094	0,0104	0,0115
4	0,0098	0,0109	0,0121	0,0134	0,0149	0,0167	0,0186	0,0071	0,0077	0,0085	0,0093	0,0103	0,0115	0,0128
5	0,0108	0,0120	0,0133	0,0148	0,0165	0,0184	0,0205	0,0077	0,0084	0,0092	0,0102	0,0114	0,0127	0,0142
6	0,0119	0,0132	0,0147	0,0163	0,0182	0,0202	0,0225	0,0084	0,0092	0,0102	0,0113	0,0126	0,0141	0,0158
7	0,0131	0,0146	0,0162	0,0180	0,0200	0,0222	0,0246	0,0091	0,0101	0,0112	0,0125	0,0140	0,0156	0,0175
8	0,0145	0,0161	0,0179	0,0198	0,0220	0,0244	0,0269	0,0100	0,0112	0,0124	0,0139	0,0155	0,0174	0,0194
9	0,0160	0,0177	0,0197	0,0218	0,0241	0,0266	0,0293	0,0111	0,0124	0,0138	0,0154	0,0172	0,0192	0,0214
10	0,0176	0,0195	0,0216	0,0239	0,0264	0,0290	0,0318	0,0123	0,0137	0,0153	0,0171	0,0191	0,0213	0,0236
11	0,0194	0,0215	0,0237	0,0261	0,0287	0,0314	0,0342	0,0136	0,0152	0,0170	0,0190	0,0211	0,0235	0,0260
12	0,0213	0,0235	0,0259	0,0284	0,0311	0,0338	0,0367	0,0151	0,0169	0,0188	0,0210	0,0233	0,0258	0,0284
13	0,0234	0,0257	0,0282	0,0308	0,0335	0,0362	0,0390	0,0168	0,0187	0,0208	0,0231	0,0256	0,0282	0,0308
14	0,0255	0,0280	0,0305	0,0332	0,0359	0,0386	0,0412	0,0186	0,0207	0,0230	0,0254	0,0279	0,0306	0,0333
15	0,0278	0,0303	0,0329	0,0355	0,0382	0,0407	0,0431	0,0206	0,0228	0,0252	0,0278	0,0304	0,0331	0,0358
16	0,0301	0,0327	0,0353	0,0378	0,0403	0,0426	0,0447	0,0227	0,0251	0,0276	0,0302	0,0328	0,0355	0,0381
17	0,0324	0,0350	0,0375	0,0399	0,0422	0,0442	0,0459	0,0249	0,0274	0,0300	0,0326	0,0352	0,0378	0,0403
18	0,0347	0,0372	0,0396	0,0418	0,0438	0,0454	0,0466	0,0273	0,0298	0,0324	0,0350	0,0375	0,0399	0,0422
19	0,0370	0,0393	0,0415	0,0434	0,0449	0,0461	0,0467	0,0296	0,0322	0,0348	0,0373	0,0397	0,0419	0,0438
20	0,0390	0,0412	0,0430	0,0445	0,0456	0,0462	0,0462	0,0320	0,0346	0,0370	0,0394	0,0415	0,0434	0,0450
21	0,0409	0,0427	0,0442	0,0452	0,0457	0,0457	0,0450	0,0344	0,0368	0,0391	0,0413	0,0431	0,0446	0,0457
22	0,0424	0,0438	0,0448	0,0453	0,0452	0,0445	0,0431	0,0366	0,0389	0,0410	0,0428	0,0443	0,0453	0,0459
23	0,0435	0,0445	0,0449	0,0448	0,0440	0,0426	0,0406	0,0387	0,0408	0,0425	0,0440	0,0450	0,0455	0,0454
24	0,0442	0,0446	0,0444	0,0436	0,0422	0,0401	0,0375	0,0405	0,0423	0,0437	0,0447	0,0452	0,0451	0,0443
25	0,0443	0,0441	0,0433	0,0418	0,0397	0,0371	0,0339	0,0421	0,0434	0,0444	0,0448	0,0447	0,0440	0,0426
26	0,0438	0,0430	0,0415	0,0394	0,0367	0,0335	0,0299	0,0432	0,0441	0,0446	0,0444	0,0437	0,0423	0,0403
27	0,0427	0,0412	0,0390	0,0363	0,0332	0,0296	0,0258	0,0439	0,0443	0,0441	0,0434	0,0420	0,0399	0,0373
28	0,0409	0,0387	0,0360	0,0329	0,0293	0,0255	0,0217	0,0440	0,0439	0,0431	0,0417	0,0396	0,0370	0,0338
29	0,0385	0,0358	0,0326	0,0290	0,0253	0,0214	0,0177	0,0436	0,0428	0,0414	0,0394	0,0367	0,0336	0,0299
30	0,0355	0,0323	0,0288	0,0250	0,0212	0,0175	0,0139	0,0426	0,0412	0,0391	0,0365	0,0333	0,0297	0,0256
31	0,0321	0,0286	0,0248	0,0210	0,0173	0,0137	0,0106	0,0409	0,0389	0,0362	0,0331	0,0295	0,0254	0,0212
32	0,0284	0,0246	0,0208	0,0171	0,0136	0,0105	0,0078	0,0387	0,0360	0,0329	0,0293	0,0252	0,0210	0,0169
33	0,0245	0,0207	0,0170	0,0135	0,0104	0,0077	0,0055	0,0358	0,0327	0,0291	0,0251	0,0209	0,0168	0,0131
34	0,0205	0,0169	0,0134	0,0103	0,0076	0,0055	0,0038	0,0325	0,0289	0,0249	0,0207	0,0167	0,0129	0,0097
35	0,0167	0,0133	0,0102	0,0075	0,0054	0,0037	0,0025	0,0287	0,0247	0,0206	0,0166	0,0129	0,0096	0,0069
36	0,0132	0,0101	0,0075	0,0053	0,0037	0,0025	0,0016	0,0246	0,0205	0,0165	0,0128	0,0095	0,0069	0,0048
37	0,0100	0,0074	0,0053	0,0037	0,0025	0,0016	0,0010	0,0204	0,0164	0,0127	0,0095	0,0068	0,0047	0,0032
38	0,0074	0,0053	0,0036	0,0024	0,0016	0,0010	0,0006	0,0163	0,0126	0,0094	0,0068	0,0047	0,0031	0,0020
39	0,0052	0,0036	0,0024	0,0016	0,0010	0,0006	0,0003	0,0125	0,0094	0,0067	0,0047	0,0031	0,0020	0,0012
40	0,0036	0,0024	0,0015	0,0010	0,0006	0,0003	0,0002	0,0093	0,0067	0,0046	0,0031	0,0020	0,0012	0,0007
41	0,0024	0,0015	0,0010	0,0006	0,0003	0,0002	0,0001	0,0067	0,0046	0,0031	0,0020	0,0012	0,0007	0,0004
42	0,0015	0,0009	0,0006	0,0003	0,0002	0,0001	0,0001	0,0046	0,0031	0,0020	0,0012	0,0007	0,0004	0,0002
43	0,0009	0,0006	0,0003	0,0002	0,0001	0,0001	0,0000	0,0030	0,0020	0,0012	0,0007	0,0004	0,0002	0,0001
44	0,0006	0,0003	0,0002	0,0001	0,0001	0,0000	0,0000	0,0019	0,0012	0,0007	0,0004	0,0002	0,0001	0,0001
45	0,0003	0,0002	0,0001	0,0001	0,0000	0,0000	0,0000	0,0012	0,0007	0,0004	0,0002	0,0001	0,0001	0,0000
46	0,0002	0,0001	0,0001	0,0000	0,0000	0,0000	0,0000	0,0007	0,0004	0,0002	0,0001	0,0001	0,0000	0,0000
47	0,0001	0,0001	0,0000	0,0000	0,0000	0,0000	0,0000	0,0004	0,0002	0,0001	0,0001	0,0000	0,0000	0,0000
48	0,0001	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0001	0,0001	0,0000	0,0000	0,0000	0,0000
49	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0001	0,0000	0,0000	0,0000	0,0000	0,0000
50	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
51	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
52	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
53	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
54	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001
55	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0000
56	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0000	0,0000
57	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000
58	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000	0,0000
59	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0000	0,0000	0,0000	0,0000	0,0000
60	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000

Table 13 - Empirical pdf for the Homogeneous Groups (gender and age) within the Portfolio

	Males			Females		
	$\mu_{m,i}(\tau)$	$\sigma_{m,i}(\tau)$	$\gamma_{m,i}(\tau)$	$\mu_{m,i}(\tau)$	$\sigma_{m,i}(\tau)$	$\gamma_{m,i}(\tau)$
60	16,37	5,56	-0,75	17,88	5,58	-0,92
61	15,88	5,54	-0,71	17,40	5,55	-0,87
62	15,38	5,51	-0,67	16,92	5,52	-0,83
63	14,88	5,48	-0,63	16,43	5,49	-0,78
64	14,37	5,45	-0,59	15,93	5,46	-0,74
65	13,87	5,41	-0,55	15,43	5,43	-0,70
66	13,36	5,37	-0,50	14,93	5,40	-0,66

Table 14 - Parameters under the Empirical pdf for the Homogeneous Groups (gender and age) within the Portfolio (Scenario 1)

	Males			Females		
	$\mu_{m,i}(\tau)$	$\sigma_{m,i}(\tau)$	$\gamma_{m,i}(\tau)$	$\mu_{m,i}(\tau)$	$\sigma_{m,i}(\tau)$	$\gamma_{m,i}(\tau)$
60	14,37	4,60	-0,97	15,54	4,52	-1,20
61	13,98	4,62	-0,91	15,18	4,52	-1,14
62	13,58	4,63	-0,85	14,81	4,53	-1,07
63	13,17	4,63	-0,79	14,42	4,54	-1,01
64	12,76	4,63	-0,73	14,03	4,55	-0,94
65	12,34	4,63	-0,67	13,63	4,56	-0,88
66	11,91	4,62	-0,62	13,22	4,57	-0,82

Table 15 - Parameters under the Empirical pdf for the Homogeneous Groups (gender and age) within the Portfolio (Scenario 2)